Matrices cuadradas no singulares totalmente no positivas y negativas: propiedades y caracterizaciones

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Definitions

$A \in \mathbb{R}^{n \times m}$

t.n. (totally negative) $\Rightarrow$ all its minors negative.

t.n.p. (totally nonpositive) $\Rightarrow$ all its minors nonpositive.

p.n. (partially negative) $\Rightarrow$ all its principal minors negative.

(N-matrix in economic models,
(Bapat-Raghavan-79)

p.n.p. (partially nonpositive) $\Rightarrow$ all its principal minors nonpositive.

Classification.

$$\{\text{t.n.}\} \subset \{\text{nonsingular t.n.p.}\} \subset \left\{ \begin{array}{c} \{\text{t.n.p.}\} \\ \{\text{p.n.}\} \end{array} \right\} \subset \{\text{p.n.p.}\}$$

Properties

Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a t.n.p. (t.n.) matrix, then

- ★ D positive diagonal matrix $\implies$ DA, AD and DAD^{-1} are t.n.p. (t.n.) matrices.
- ★ All submatrix of A is a t.n.p. (t.n.) matrix.
- ★ If P is a permutation matrix then, in general, PA, AP and PAP are not t.n.p. (t.n.) matrices.
- ★ If P is the permutation matrix $[n,\ n-1,\ \ldots, 2,\ 1]$ $\implies$ PAP is a t.n.p. (t.n.) matrix.
- ★ If A is nonsingular $\implies$ $a_{ij} < 0,\ \forall i, j = 1, 2, \ldots, n$, except for a_{11} and a_{nn}.
- ★ If $a_{ii} = 0$, for $1 < i < n$ $\implies$ a_i is a zero-column or $a^{(i)}$ is a zero-row.

Problem to study

$$A = (a_{ij}) \in \mathbb{R}^{n \times m} \text{ t.n.p. (t.n.)}$$
$$\Downarrow$$

Characterize A through the LDU factorization or the bidiagonal factorization using the Neville o quasi-Neville elimination process without pivoting.

We can consider the following cases:

- A t.n.p. (t.n.) matrix with $a_{11} < 0$ $\longrightarrow$
  - A nonsingular
  - A singular
  - A rectangular
- A t.n.p. matrix with $a_{11} = 0$ $\longrightarrow$
  - A nonsingular
  - A singular
  - A rectangular

Characterization of t.n. matrices

[M. Gasca - J.M. Peña, A test for strict sign-regularity, LAA, 1994]
[S. Fallat - V. Driessche, On matrices with all minors negative,ELA, 2000]

Theorem

$A = (a_{ij}) \in \mathbb{R}^{n\times n}$, with $a_{nn} < 0$. Then,

$$A \text{ is t.n.}$$
$$\Updownarrow$$
$$A = LDU$$

L (U) is a unit lower (upper) ΔSTP matrix
$D = \text{diag}(-d_1, d_2, d_3, \ldots, d_n)$ with $d_i > 0$, for $i = 1, 2, \ldots, n$.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 4/5 & 1 & 0 \\ 1/5 & 2/3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -20 & 0 & 0 \\ 0 & 3/5 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 4/5 & 1/5 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

$$= \begin{bmatrix} -20 & -16 & -4 \\ -16 & -12.2 & -2.8 \\ -4 & -2.8 & -0.2 \end{bmatrix} = A$$

Relations

[Fallat-Van Den Driessche-ELA2000]:

A relation between TP (STP) and t.n.p. (t.n.) matrices is

"the product of two t.n.p. (t.n.) matrices is a TP (STP) matrix"

Semi-open Problem

"Can any $n \times n$ TP (STP) matrix be written
as the product of two t.n.p. (t.n.) matrices?"

- If $n = 2$, YES.
- If $n \geq 3$, in general NO

The upper (lower) triangular, nonsingular TP (STP) matrix of size $n \times n$, $n \geq 3$, can not be written as the product of two t.n.p. (t.n.) matrices.

Characterization of t.n.p. matrices. Necessary condition

Theorem 1. [Cantó, Koev, Ricarte, U. - 2008]

$$A \in \mathbb{R}^{n\times n}, \text{ t.n.p., nonsingular, } a_{11} < 0$$

$$\Downarrow$$

$$A = LDU$$

- L is a unit lower triangular TP matrix.
- U is a unit upper triangular TP matrix.
- $D = \text{diag}(-d_1, d_2, \ldots, d_n)$ with $d_i > 0$, for $i = 1, 2, \ldots, n$.

Sketch of the proof Th. 1

$$A \text{ nonsingular t.n.p. } \; a_{11} < 0$$
$$\Downarrow$$
$$(P = [n, n-1, \ldots, 1])$$
$$\Downarrow$$
$$G = PAP \;\; \text{nonsingular t.n.p. } \; g_{nn} = a_{11} < 0$$
$$\Downarrow$$
$$(S = \mathrm{diag}(1, -1, \ldots, (-1)^{n+1}))$$
$$\Downarrow$$
$$B = SG^{-1}S$$

For $\forall \alpha, \beta \in \mathcal{Q}_{k,n}, \quad k = 1, 2, \ldots, n-1,$

$$\det B[\alpha|\beta] = \det SG^{-1}S[\alpha|\beta] = \frac{\det G[\alpha^c|\beta^c]}{\det G} \geq 0,$$

since $\det(B) < 0$, then

$$B \;\; \text{is sign-regular with signature } \; \epsilon = (1, \ldots, 1, -1)$$

Sketch of the proof Th. 1

For $\boxed{x > 0}$ construct the matrix

$$C = B + xE_{nn} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n-1} & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n-1} & b_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ b_{n-11} & b_{n-12} & \cdots & b_{n-1n-1} & b_{n-1n} \\ b_{n1} & b_{n2} & \cdots & b_{nn-1} & b_{nn} + x \end{bmatrix}$$

verifies

- $\forall \alpha,\ \beta \in \mathcal{Q}_{k,n},\ k = 1, 2, \ldots, n-1,\ n \notin \alpha \cap \beta$

$$\det C[\alpha \,|\, \beta] = \det B[\alpha \,|\, \beta] \geq 0$$

- $\forall \alpha,\ \beta \in \mathcal{Q}_{k,n},\ k = 1, 2, \ldots, n-2,$

$$\det C[\alpha, n \,|\, \beta, n] = \det B[\alpha, n \,|\, \beta, n] + x \det B[\alpha \,|\, \beta] \geq 0$$

Sketch of the proof Th. 1

- For $k = n$,

$$\begin{aligned} \det C &= \det B + x \det B[1,2,\ldots,n-1] = \det B + x \det SG^{-1}S[1,2,\ldots,n-1] \\ &= \det B + x\frac{\det G[n]}{\det G} = \det B + x\frac{a_{11}}{\det A} = \frac{1}{\det A} + x\,\frac{a_{11}}{\det A} \end{aligned}$$

Since $a_{11} < 0$, if we choose $\quad x > -\dfrac{1}{a_{11}} > 0 \implies \quad \det C > 0$

Then,

$$C \quad \text{is TP}$$
$$\Downarrow$$
$$C = L'D'U' \quad \begin{cases} L' & \text{is unit lower triangular TP matrix} \\ U' & \text{is unit upper triangular TP matrix} \\ D' = \operatorname{diag}(d_1', d_2', \ldots, d_n'), & d_i' > 0 \quad i = 1,2,\ldots,n \end{cases}$$

Sketch of the proof Th. 1

Consider

$$B = C - xE_{nn} = L'D'U' - xE_{nn}$$

$$= \begin{bmatrix} L_1' & O \\ l_1' & 1 \end{bmatrix} \begin{bmatrix} D_1' & 0 \\ 0 & d_{nn}' \end{bmatrix} \begin{bmatrix} U_1' & u_1' \\ O & 1 \end{bmatrix} - x \begin{bmatrix} O & O \\ O & 1 \end{bmatrix}$$

$$= \begin{bmatrix} L_1' & O \\ l_1' & 1 \end{bmatrix} \begin{bmatrix} D_1' & O \\ O & d_{nn}' - x \end{bmatrix} \begin{bmatrix} U_1' & u_1' \\ O & 1 \end{bmatrix} = L'D''U'$$

Since $\det B < 0 \implies \boxed{d_{nn}' - x < 0}$. Then, we have

$$\begin{aligned} A &= PGP = P(SB^{-1}S)P = P(S(L'D''U')^{-1}S)P \\ &= PS(U')^{-1}(D'')^{-1}(L')^{-1}SP = \\ &= \underbrace{[PS(U')^{-1}SP]}_{L}\underbrace{[PS(D'')^{-1}SP]}_{D}\underbrace{[PS(L')^{-1}SP]}_{U} \end{aligned}$$

$$A = LDU \quad \begin{cases} L: & \text{unit lower triangular TP matrix} \\ U: & \text{unit upper triangular TP matrix} \\ D = \operatorname{diag}(-d_1, d_2, \ldots, d_n), & d_i > 0 \quad i = 1, 2, \ldots, n \end{cases}$$

Example

Consider the following t.n.p. matrix

$$A = \begin{bmatrix} -4 & -8 & -16 & -16 \\ -16 & -31 & -59 & -56 \\ -80 & -153 & -283 & -258 \\ -80 & -147 & -245 & -185 \end{bmatrix}$$

By Neville elimination we have,

- $E_2^{(1)}(-4)E_3^{(1)}(-5)E_4^{(1)}(-1)A = \begin{bmatrix} -4 & -8 & -16 & -16 \\ 0 & 1 & 5 & 8 \\ 0 & 2 & 12 & 22 \\ 0 & 6 & 38 & 73 \end{bmatrix} = A_1$

- $E_3^{(2)}(-2)E_4^{(2)}(-3)A_1 = \begin{bmatrix} -4 & -8 & -16 & -16 \\ 0 & 1 & 5 & 8 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 2 & 7 \end{bmatrix} = A_2$

Example

- $E_4^{(3)}(-1)A_2 = \begin{bmatrix} -4 & -8 & -16 & -16 \\ 0 & 1 & 5 & 8 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} = A_3$

Then,

$$E_4^{(3)}(-1)E_3^{(2)}(-2)E_4^{(2)}(-3)E_2^{(1)}(-4)E_3^{(1)}(-5)E_4^{(1)}(-1)A = A_3$$
$$\Downarrow$$
$$A = \underbrace{E_4^{(1)}(1)E_3^{(1)}(5)E_2^{(1)}(4)}_{E^{(1)}}\underbrace{E_4^{(2)}(3)E_3^{(2)}(2)}_{E^{(2)}}\underbrace{E_4^{(3)}(1)}_{E^{(3)}}A_3$$

where

$$E^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 20 & 5 & 1 & 0 \\ 20 & 5 & 1 & 1 \end{bmatrix}, \quad E^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 6 & 3 & 1 \end{bmatrix}, \quad E^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Example

$$A_3 F_4^{(1)}(-1) F_3^{(1)}(-2) F_2^{(1)}(-2) F_4^{(2)}(-1) F_3^{(2)}(-3) F_4^{(3)}(-1) = \underbrace{\begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{D}$$

$$\Downarrow$$

$$A_3 = D \underbrace{F_4^{(3)}(1)}_{F^{(3)}} \underbrace{F_3^{(2)}(3) F_4^{(2)}(1)}_{F^{(2)}} \underbrace{F_2^{(1)}(2) F_3^{(1)}(2) F_4^{(1)}(1)}_{F^{(1)}}$$

where

$$F^{(1)} = \begin{bmatrix} 1 & 2 & 4 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad F^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad F^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

Therefore, we have

$$A = \underbrace{E^{(1)}E^{(2)}E^{(3)}}_{L}\,D\,\underbrace{F^{(3)}F^{(2)}F^{(1)}}_{U}$$

where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 20 & 7 & 1 & 0 \\ 20 & 13 & 5 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 4 & 4 \\ 0 & 1 & 5 & 8 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Remark

$$A \in \mathbb{R}^{n\times n},\ \text{t.n.p., nonsingular, } a_{11} < 0$$

$$\Downarrow$$

$$A = LDU$$

Then, for $i, j = 1, 2, \dots, n$,

- $l_{i1} > 0, \quad u_{1j} > 0$

$$(a_{ij} = l^{(i)} D u_j = -d_1 l_{i1} u_{1j} + d_2 l_{i2} u_{2j} + \cdots + d_n l_{in} u_{nj} = -d_1 l_{i1} u_{1j} + K, \quad K > 0)$$

- Since L is TP, for all $i > j$, $j = 1, 2, \dots, n-1$, we have

$$\det L[j, i|1, j] = \det \begin{bmatrix} l_{j1} & 1 \\ l_{i1} & l_{ij} \end{bmatrix} = l_{j1} l_{ij} - l_{i1} \geq 0 \longrightarrow \boxed{l_{ij} > 0}$$

- Since U is TP, for all $i > j$, $j = 1, 2, \dots, n-1$, we have

$$\det U[1, j|j, i] \det \begin{bmatrix} u_{1j} & u_{1i} \\ 1 & u_{ji} \end{bmatrix} = u_{1j} u_{ji} - u_{1i} \geq 0 \longrightarrow \boxed{u_{ji} > 0}$$

Characterization of t.n.p. matrices

The converse is not true in general, the matrix

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

$$= \begin{bmatrix} -3 & -6 & -12 \\ -6 & -10 & -20 \\ -3 & -2 & 1 \end{bmatrix}$$

is not a t.n.p. matrix because $a_{33} = 1 > 0$.

Characterization of t.n.p. matrices. Sufficient condition

Theorem 2.

$A = LDU \in \mathbb{R}^{n\times n}$ with:

- $a_{nn} \leq 0$
- L: unit lower triangular TP matrix, $l_{ij} > 0$, $\forall i > j$
- U: unit upper triangular TP matrix, $u_{ji} > 0$, $\forall i > j$
- $D = \text{diag}(-d_1, d_2, \ldots, d_n)$, $d_i > 0$, $i = 1, 2, \ldots, n$

$$\Downarrow$$

A is t.n.p.

Sketch of the proof Th. 2

$\boxed{a_{nn} < 0}$

$$L = (E_n(m_{n1}) \cdots E_2(m_{21}))(E_n(m_{n2}) \cdots E_3(m_{32})) \cdots E_n(m_{n,n-1})$$

$m_{ij} \geq 0$ multipliers of the Neville elimination

If $m_{ij} = 0 \;\Rightarrow\; m_{ij} = \delta > 0 \;\Rightarrow\; L(\delta)$ unit lower $\triangle$STP

$$A(\delta) = L(\delta)DU(\delta) = A + \left[\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & p_{ij}(\delta) & \\ 0 & & & \end{array}\right]$$

$$\lim_{\delta \to 0} p_{ij}(\delta) = 0$$

$$\exists \delta_0 \mid \forall \delta > \delta_0 \quad A(\delta)(n,n) = a_{nn} + p_{nn}(\delta) < 0 \Rightarrow A(\delta) \text{ is t.n.}$$

$$\det A[\alpha|\beta] = \lim_{\delta \to 0} \det A(\delta)[\alpha|\beta] \leq 0 \;\Rightarrow\; A \text{ is t.n.p.}$$

Sketch of the proof Th. 2

$\boxed{a_{nn} = 0}$

$$B = A - xE_{nn} = LDU - xE_{nn} = L \begin{bmatrix} D_{n-1} & \\ & d_n - x \end{bmatrix} U = LD'U$$

$$\left.\begin{array}{l} b_{nn} = -x < 0 \quad \forall x > 0 \\ \forall x \mid 0 < x < d_n \;\Rightarrow\; d'_{nn} = d_n - x > 0 \end{array}\right\} \Rightarrow B \text{ is t.n.p.}$$

$$\det A[\alpha|\beta] = \det B[\alpha|\beta] \leq 0 \qquad n \notin \alpha \cap \beta$$

$$\det A[\alpha, n|\beta, n] = \det B[\alpha, n|\beta, n] + x \det B[\alpha|\beta] \leq 0$$

$$\Downarrow$$

A is t.n.p.

Characterization of t.n.p. and t.p. matrices

Theorem 3.

$$A \in \mathbb{R}^{n\times n} \text{ nonsingular with } a_{11} < 0, a_{nn} \leq 0 \text{ is t.n.p.}$$

$$\Updownarrow$$

$$A = LDU \quad \begin{cases} L \text{ is unit lower triangular TP matrix, } l_{ij} > 0, \quad \forall i > j \\ U \text{ is unit upper triangular TP matrix, } u_{ji} > 0 \quad \forall i > j \\ D = \operatorname{diag}(-d_1, d_2, \ldots, d_n), \quad d_i > 0 \quad i = 1, 2, \ldots, n \end{cases}$$

Question

$$A \text{ nonsingular t.n.p. } \; a_{11} < 0$$
$$\Updownarrow$$
$$(P = [n, n-1, \dots, 1])$$
$$\Updownarrow$$
$$G = PAP \text{ nonsingular t.n.p. } \; g_{nn} = a_{11} < 0$$
$$\Updownarrow$$
$$(S = \text{diag}(1, -1, \dots, (-1)^{n+1}))$$
$$\Updownarrow$$
$$B = SG^{-1}S, \;\; \epsilon(B) = (1, 1, \dots, 1, -1)$$
$$\Downarrow$$
$$\exists \; x > 0 \; : \; C_x = B + xE_{nn} \;\text{ is TP}$$

$$C \text{ nonsingular TP}$$
$$\Downarrow \;\; ?$$
$$\exists \; x > 0 \; : \; A_x = PS(C - xE_{nn})^{-1}SP \text{ is t.n.p.}$$

Example

Consider the lower, upper and symmetric Pascal TP matrices of size 5×5

$$L_{1,5} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}, \quad U_{1,5} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 & 4 \\ & & 1 & 3 & 6 \\ & & & 1 & 4 \\ & & & & 1 \end{bmatrix} = L_{1,5}^T$$

$$P_{1,5} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix} = L_{1,5} L_{1,5}^T$$

Question:

$$\exists x > 0 \; : \; A_x = PS \left(P_{1,5} - xE_{55} \right)^{-1} SP \quad \text{is t.n.p.?}$$

Example

With $\boxed{x > 0}$ construct the matrix

$$\begin{aligned} B_x &= P_{1,5} - xE_{55} = L_{1,5}L_{1,5}^T - xE_{55} = L_{1,5}\left(I - xE_{55}\right)L_{1,5}^T \\ &= L_{1,5}\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1-x \end{bmatrix} L_{1,5}^T = L_{1,5}\,\left(I_{5,1-x}\right)\,L_{1,5}^T \end{aligned}$$

Then, $\det B = 1 - x < 0 \longrightarrow \boxed{1 < x}$

$$\begin{aligned} G_x &= SB_x^{-1}S = S(L_{1,5}\,\left(I_{5,1-x}\right)\,L_{1,5}^T)^{-1}S = SL_{1,5}^{-T}\,\left(I_{5,1-x}\right)^{-1}\,L_{1,5}^{-1}S \\ &= \left(SL_{1,5}^{-T}S\right)\left(SI_{5,1/1-x}S\right)\left(SL_{1,5}^{-1}S\right) = L_{1,5}^T\,I_{5,1/1-x}\,L_{1,5} \end{aligned}$$

Example

$$A_x = PG_xP = P\left(L_{1,5}^T\, I_{5,1/1-x}\, L_{1,5}\right)P = \left(PL_{1,5}^T P\right)\left(PI_{5,1/1-x}P\right)\left(PL_{1,5}P\right)$$

$$= \underbrace{\begin{bmatrix} 1 & & & & \\ 4 & 1 & & & \\ 6 & 3 & 1 & & \\ 4 & 3 & 2 & 1 & \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}}_{L_{5,5}} \underbrace{\begin{bmatrix} \frac{1}{1-x} & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}}_{I_{1,1/1-x}} \underbrace{\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ & 1 & 3 & 3 & 1 \\ & & 1 & 2 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}}_{L_{5,5}^T}$$

$$A_x \text{ t.n.p.} \iff A_x(n,n) = \frac{1}{1-x} + 4 \leq 0 \longrightarrow \boxed{x \leq \frac{5}{4}}$$

Therefore,

$$\boxed{\forall x \in \left]1, \tfrac{5}{4}\right] \text{ the matrix } A_x \text{ is t.n.p.}}$$

Counterexample

Consider the matrix of size $n \times n$

$$
\begin{aligned}
C_n &= [\min\{i,j\}] = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & n-1 & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 1 & 1 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ 1 & 1 & 1 & \cdots & 1 & \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ & 1 & 1 & \cdots & 1 & 1 \\ & & 1 & \cdots & 1 & 1 \\ & & & \ddots & \vdots & \vdots \\ & & & & 1 & 1 \\ & & & & & 1 \end{bmatrix}}_{L^T}
\end{aligned}
$$

Counterexample

With $\boxed{x > 0}$ construct the matrix

$$B_x = C_n - xE_{nn} = L \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1-x \end{bmatrix} L^T = L\left(I_{n,1-x}\right)L^T$$

we need that $\epsilon(B_x) = (1, 1, \dots, 1, -1)$.

Since

$$\det B_x = 1 - x < 0 \longrightarrow \boxed{1 < x}$$

$$\Downarrow$$

$$\det B_x[n-1, n] = \det \begin{bmatrix} n-1 & n-1 \\ n-1 & n-x \end{bmatrix} = (n-1)(1-x) < 0$$

Therefore $\nexists x \ : \ \epsilon(B_x) = (1, 1, \dots, 1, -1)$

Contraejemplo

Construimos

$$B_x = C_n - xE_{nn} = L\ (I_{n,1-x})\ L^T$$
$$\downarrow$$
$$G_x^{-1} = SB_xS$$
$$\downarrow$$
$$G_x = SB_x^{-1}S = S\left(L^{-T}\left(I_{n,\frac{1}{1-x}}\right)L^{-1}\right)S$$
$$= \left(SL^{-T}S\right)\left(SI_{n,\frac{1}{1-x}}S\right)\left(SL^{-1}S\right)$$
$$\downarrow$$
$$A_x = PG_xP = \left(PSL^{-T}SP\right)\left(PSI_{n,\frac{1}{1-x}}SP\right)\left(PSL^{-1}SP\right) =$$

$$\underbrace{\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & 1 & 1 \end{bmatrix}}_{L_{A_x}} \underbrace{\begin{bmatrix} \frac{1}{1-x} & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}}_{D_{A_x}} \underbrace{\begin{bmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ & & & & & 1 \end{bmatrix}}_{U_{A_x}=L_{A_x}^T}$$

Characterization by minors

Theorem 4.

$A \in \mathbb{R}^{n\times n}$ nonsingular with all entries negative except $a_{nn} \leq 0$.

$$A \quad \text{t.n.p.} \quad \Longleftrightarrow \begin{cases} \det A[\alpha|1,2,\ldots,k] \leq 0, & \forall \alpha \in \mathcal{Q}_{k,n} \\ \det A[1,2,\ldots,k|\beta] \leq 0, & \forall \beta \in \mathcal{Q}_{k,n} \\ \det A[1,2,\ldots,k] < 0 \end{cases}$$
$$k = 1,2,\ldots,n$$

Theorem 5.

$A \in \mathbb{R}^{n\times n}$ nonsingular with all entries negative

$$A \quad \text{t.n.} \quad \Longleftrightarrow \begin{cases} \det A[\alpha|1,2,\ldots,k] < 0, & d(\alpha) = 0 \\ \det A[1,2,\ldots,k|\beta] < 0, & d(\beta) = 0 \end{cases}$$
$$k = 1,2,\ldots,n$$

Bidiagonal characterization of t.n.p. matrices

Consider the following t.n.p. matrix

$$A = \begin{bmatrix} -4 & -8 & -16 & -16 \\ -16 & -31 & -59 & -56 \\ -80 & -153 & -283 & -258 \\ -80 & -147 & -245 & -185 \end{bmatrix}$$

We have seen that

$$\begin{aligned} A &= E_4^{(1)}(1)E_3^{(1)}(5)E_2^{(1)}(4)E_4^{(2)}(3)E_3^{(2)}(2)E_4^{(3)}(1)A_3 \\ &= \underbrace{E_4^{(1)}(1)}_{F_3}\underbrace{E_3^{(1)}(5)E_4^{(2)}(3)}_{F_2}\underbrace{E_2^{(1)}(4)E_3^{(2)}(2)E_4^{(3)}(1)}_{F_1}A_3 \end{aligned}$$

where

$$F_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Bidiagonal characterization of t.n.p. matrices

$$A_3 = DF_4^{(3)}(1)F_3^{(2)}(3)F_4^{(2)}(1)F_2^{(1)}(2)F_3^{(1)}(2)F_4^{(1)}(1)$$

$$= D\underbrace{F_4^{(3)}(1)F_3^{(2)}(3)F_2^{(1)}(2)}_{G_1}\underbrace{F_4^{(2)}(1)F_3^{(1)}(2)}_{G_2}\underbrace{F_4^{(1)}(1)}_{G_3}$$

where

$$G_1 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Bidiagonal characterization of t.n.p. matrices

Then,

$$A = F_3 F_2 F_1 D G_1 G_2 G_3$$

where F_1, F_2, F_3, G_1, G_2 y G_3 are bidiagonal matrices.

Theorem 6.

$$\text{If} \quad A = (a_{ij}) \in \mathbb{R}^{n\times n} \quad \text{nonsingular t.n.p. matrix with} \quad a_{11} < 0$$

$$\Downarrow$$

$$A = F_{n-1} F_{n-2} \ldots F_1 D G_1 G_2 \ldots G_{n-1}$$

- $D = \text{diag}(-d_1, d_2, \ldots, d_n)$, with $d_i > 0$,
- F_i for $i = 1, 2, \ldots, n-1$, are unit lower bidiagonal TP matrices,
- G_i for $i = 1, 2, \ldots, n-1$, are unit upper bidiagonal TP matrices.

Bidiagonal characterization of t.n.p. matrices

For $j = 1, 2, \ldots, n-1$

$$F_{n-j} = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & \beta_{1,n-j+1} & 1 & & & \\ & & & \ddots & \ddots & & \\ & & & & \beta_{j-1,n-1} & 1 & \\ & & & & & \beta_{j,n} & 1 \end{bmatrix},$$

- $\beta_{1,2} > 0, \ \ldots, \ \beta_{1,n} > 0$
- If $\beta_{s,t} = 0$ then $\beta_{s,h} = 0$, for all $h > t$,

Bidiagonal characterization of t.n.p. matrices

For $k = 1, \ldots, n-1$

$$G_k = \begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & \alpha_{1,k+1} & & & & \\ & & & & \ddots & \ddots & & & \\ & & & & & 1 & \alpha_{n-k-1,n-1} & & \\ & & & & & & 1 & \alpha_{n-k,n} & \\ & & & & & & & 1 & \end{bmatrix},$$

- $\alpha_{1,2} > 0, \ldots, \alpha_{1,n} > 0$,
- If $\alpha_{s,t} = 0$ then $\alpha_{s,h} = 0$, for all $h > t$.

Bidiagonal characterization of t.n.p. matrices

$$A = F_3 F_2 F_1 D G_1 G_2 G_3$$

where

$$F_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and

$$G_1 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From this factorization we have

$$\mathcal{BD}_{(t.n.p.)}(A) = \begin{bmatrix} -4 & 2 & 2 & 1 \\ 4 & 1 & 3 & 1 \\ 5 & 2 & 2 & 1 \\ 1 & 3 & 1 & 1 \end{bmatrix}$$

Bidiagonal characterization of t.n.p. matrices

Definition 1

$A \in \mathbb{R}^{n\times n}$ nonsingular t.n.p. with $a_{11} < 0$.
We store the entries of the bidiagonal factorization of A in the following matrix

$$\mathcal{BD}_{(t.n.p.)}(A) = \begin{bmatrix} -d_1 & \alpha_{1,2} & \alpha_{1,3} & \cdots & \alpha_{1,n-1} & \alpha_{1,n} \\ \beta_{1,2} & d_2 & \alpha_{2,3} & \cdots & \alpha_{2,n-1} & \alpha_{2,n} \\ \beta_{1,3} & \beta_{2,3} & d_3 & \cdots & \alpha_{3,n-1} & \alpha_{3,n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \beta_{1,n-1} & \beta_{2,n-1} & \beta_{3,n-1} & \cdots & d_{n-1} & \alpha_{n-1,n} \\ \beta_{1,n} & \beta_{2,n} & \beta_{3,n} & \cdots & \beta_{n-1,n} & d_n \end{bmatrix}$$

where

$$\begin{array}{rcl} d_i & > & 0, \quad \beta_{i,j} \geq 0, \quad \alpha_{i,j} \geq 0 \quad \text{for} \quad i,j = 1,2,\ldots,n, \\ \beta_{1,j} & > & 0, \quad \alpha_{1,j} > 0 \quad \text{for} \quad j = 2,3,\ldots,n, \\ \text{if} \quad \beta_{i,j} & = & 0, \quad (\alpha_{i,j} = 0) \implies \beta_{i,h} = 0, \quad (\alpha_{i,h} = 0) \text{ for all } \ h > j. \end{array}$$

Counterexample

Consider the following matrix

$$\mathcal{BD}_{(t.n.p.)}(A) = \begin{bmatrix} -3 & 2 & 2 & 1 & 2 \\ 4 & 1 & 3 & 1 & 1 \\ 5 & 2 & 2 & 1 & 0 \\ 1 & 3 & 1 & 1 & 1 \\ 2 & 3 & 0 & 2 & 1 \end{bmatrix}.$$

Constructing the corresponding bidiagonal matrices and multiplying we obtain

$$A = \begin{bmatrix} -3 & -6 & -12 & -12 & -24 \\ -12 & -23 & -43 & -40 & -77 \\ -60 & -113 & -203 & -178 & -331 \\ -60 & -107 & -165 & -105 & -149 \\ -120 & -196 & -216 & 11 & 257 \end{bmatrix},$$

which is not a t.n.p. matrix because the $(5,4)$ and $(5,5)$ entries are positive.

Bidiagonal characterization of t.n.p. matrices

Theorem 7.

Let $\mathcal{BD}_{(t.n.p.)}(A)$ be a matrix satisfying conditions of Definition 1 and let $A \in \mathbb{R}^{n\times n}$ the obtained matrix from $\mathcal{BD}_{(t.n.p.)}(A)$, with $a_{11} < 0$.

Then, A is a t.n.p. matrix if $a_{nn} \leq 0$.

EXAMPLE

The matrix

$$BD_{(t.n.p.)}(\mathrm{A}) = \begin{pmatrix} -4 & 2 & 2 & 1 & 2 \\ 4 & 1 & 3 & 1 & 1 \\ 5 & 2 & 2 & 1 & 0 \\ 1 & 3 & 1 & 1 & 1 \\ 2 & 3 & 0 & 2 & 1 \end{pmatrix}$$

is the bidiagonal factorization of a t.n.p. matrix?

We need to know a_{55}

$$A = F_4 F_3 F_2 F_1 \text{ D } G_1 G_2 G_3 G_4 = \text{L D U}$$

$$\Downarrow$$

$$a_{55} = (l_{51}\ \ l_{52}\ \ l_{53}\ \ l_{54}\ \ 1)\ \text{D} \begin{pmatrix} u_{15} \\ u_{25} \\ u_{35} \\ u_{45} \\ 1 \end{pmatrix}$$

We need to know a_{55}

$$A = F_4 F_3 F_2 F_1 \ D \ G_1 G_2 G_3 G_4 = L \ D \ U$$

⇓

$$a_{55} = (l_{51} \ l_{52} \ l_{53} \ l_{54} \ 1) \ D \begin{pmatrix} u_{15} \\ u_{25} \\ u_{35} \\ u_{45} \\ 1 \end{pmatrix}$$

We need to know a_{55}

$$A = F_4 F_3 F_2 F_1 \; D \; G_1 G_2 G_3 G_4 = L \; D \; U$$

⇓

$$a_{55} = \begin{pmatrix} l_{51} & l_{52} & l_{53} & l_{54} & 1 \end{pmatrix} D \begin{pmatrix} u_{15} \\ u_{25} \\ u_{35} \\ u_{45} \\ 1 \end{pmatrix}$$

We need to know a_{55}

$$a_{55} = (l_{51}\ l_{52}\ l_{53}\ l_{54}\ 1)\ D \begin{pmatrix} u_{15} \\ u_{25} \\ u_{35} \\ u_{45} \\ 1 \end{pmatrix}$$

$\boldsymbol{a_{55}} \leq \mathbf{0}$ YES

$\boldsymbol{a_{55}} > \mathbf{0}$ NO

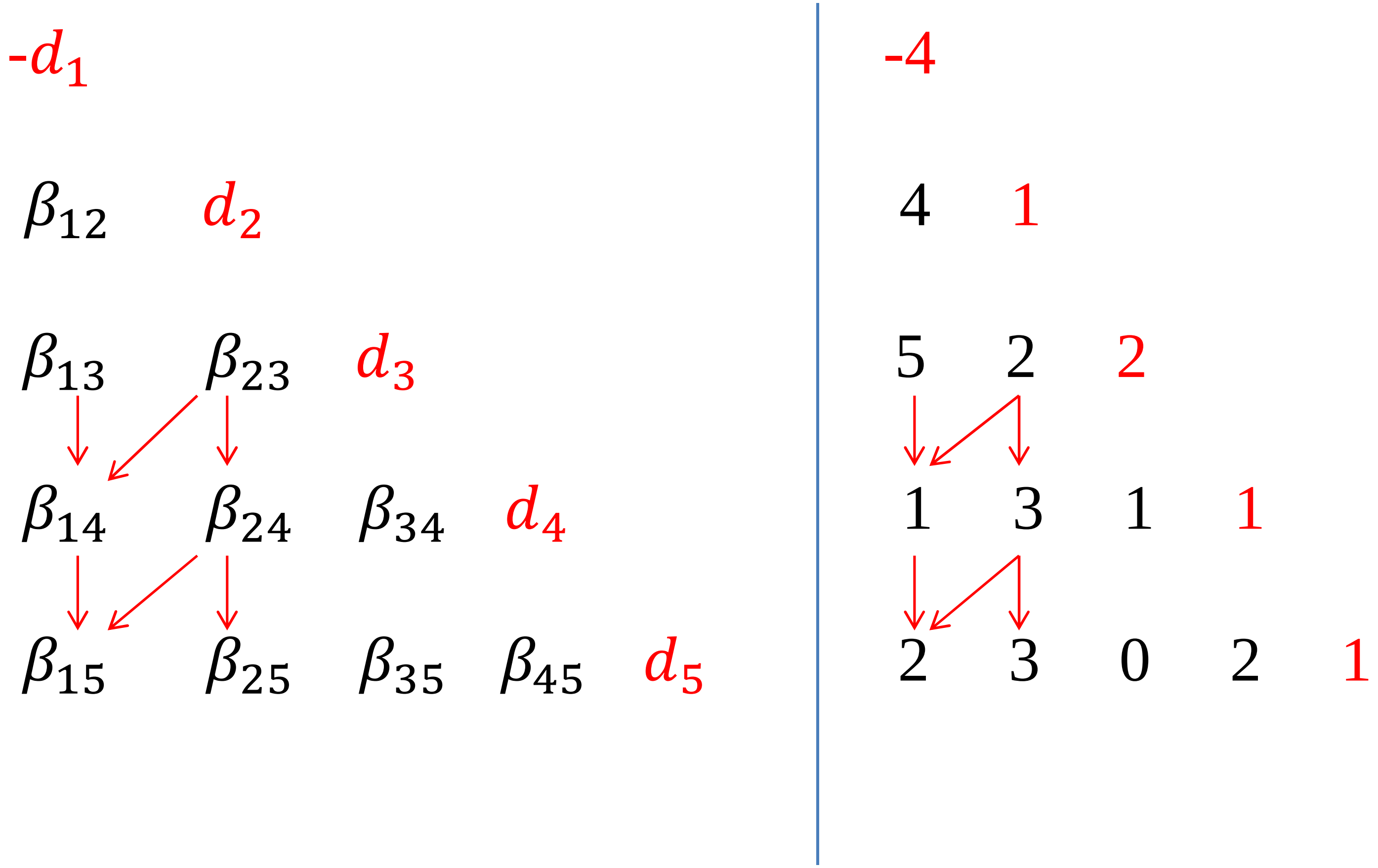
$BD_{(t.n.p.)}$(A) is the bidiagonal factorization of a t.n.p. matrix

$-d_1$					-4					
β_{12}	d_2				4	1				
β_{13}	β_{23}	d_3			5	2	2			
β_{14}	β_{24}	β_{34}	d_4		1	3	1	1		
β_{15}	β_{25}	β_{35}	β_{45}	d_5	2	3	0	2	1	

$-d_1$				
β_{12}	d_2			
↓				
β_{13}	β_{23}	d_3		
↓				
β_{14}	β_{24}	β_{34}	d_4	
↓				
β_{15}	β_{25}	β_{35}	β_{45}	d_5

-4				
4	1			
↓				
5	2	2		
↓				
1	3	1	1	
↓				
2	3	0	2	1

$$l_{51} = \beta_{12}\,\beta_{13}\,\beta_{14}\,\beta_{15}$$
$$= 4\,.\,5\,.\,1\,.\,2 = 40$$



$$l_{52} = \beta_{13}\,\beta_{14}\,\beta_{15} + \beta_{23}\,\beta_{14}\,\beta_{15} + \beta_{23}\,\beta_{24}\,\beta_{15} + \beta_{23}\,\beta_{24}\,\beta_{25}$$
$$= 5 \,.\, 1 \,.\, 2 + 2 \,.\, 1 \,.\, 2 + 2 \,.\, 3 \,.\, 2 + 2 \,.\, 3 \,.\, 3 = 44$$

$-d_1$				
β_{12}	d_2			
β_{13}	β_{23}	d_3		
β_{14}	β_{24}	β_{34}	d_4	
β_{15}	β_{25}	β_{35}	β_{45}	d_5

-4				
4	1			
5	2	2		
1	3	1	1	
2	3	0	2	1

$$l_{53} = \beta_{14}\,\beta_{15} + \beta_{24}\,\beta_{15} + \beta_{24}\,\beta_{25} + \beta_{34}\,\beta_{15} + \beta_{34}\,\beta_{25} + \beta_{34}\,\beta_{35}$$
$$= 1\,.\,2 + 3\,.\,2 + 3\,.\,3 + 1\,.\,2 + 1.\,3 = 22$$

$$l_{54} = \beta_{15} + \beta_{25} + \beta_{35} + \beta_{45} = 2 + 3 + 0 + 2 = 7$$

Then

$$a_{55=}\begin{pmatrix}40 & 35 & 22 & 7 & 1\end{pmatrix} \mathrm{D} \begin{pmatrix}u_{15} \\ u_{25} \\ u_{35} \\ u_{45} \\ 1\end{pmatrix}$$

Then

$$a_{55} = \begin{pmatrix}40 & 35 & 22 & 7 & 1\end{pmatrix} \mathrm{D} \begin{pmatrix}u_{15}\\ u_{25}\\ u_{35}\\ u_{45}\\ 1\end{pmatrix} =$$

$$\begin{pmatrix}40 & 35 & 22 & 7 & 1\end{pmatrix} \mathrm{D} \begin{pmatrix}8\\ 19\\ 8\\ 4\\ 1\end{pmatrix} = -63 < 0$$

$BD_{(t.n.p.)}$(A) is the bidiagonal factorization of a t.n.p. matrix

The t.n.p. matrix is

$$\mathbf{A}=\begin{pmatrix} -4 & -8 & -16 & -16 & -32 \\ -16 & -31 & -59 & -56 & -109 \\ -80 & -153 & -283 & -258 & -491 \\ -80 & -147 & -245 & -185 & -309 \\ -160 & -276 & -376 & -149 & -63 \end{pmatrix}$$