Total Positivity and related classes of matrices

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STP, TP, SSR, SR, TN and TNP matrices

Definition. A matrix is *strictly totally positive* (STP) if all its minors are positive and it is *totally positive* (TP) if all its minors are nonnegative.

Definition. A matrix is called *sign-regular* (SR) if all $k \times k$ minors of A have the same sign (which may depend on k) for all k. If, in addition, all minors are nonzero, then it is called *strictly sign-regular* (SSR).

Variation diminishing properties of sign-regular matrices A: if A is a nonsingular $(n+1) \times (n+1)$ matrix, then A is sign-regular if and only if the number of changes of strict sign in the ordered sequence of components of $A\mathbf{x}$ is less than or equal to the number of changes of strict sign in the ordered sequence $(x_0, \ldots, x_n)$, for all $\mathbf{x} = (x_0, \ldots, x_n)^T \in \mathbf{R}^{n+1}$.

I.J. Schoenberg: Über Variationsderminderende lineare Transformationem. *Math. Z.* **32** (1930), 321-328.

Definition. A matrix is *totally negative* (TN) if all its minors are negative and it is *totally nonpositive* (TP) if all its minors are nonpositive

Books on Total Positivity

- F.R. Gantmacher, M.G. Krein: *Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme.* Akademie-Verlag, Berlin, 1960. Revised English version: *Oscillation matrices*, AMS, Providence, 2002.
- S. Karlin: *Total Positivity*, Vol. I, Stanford University Press, Calif., 1968.
- M. Gasca, C.A. Micchelli (Eds.): *Total positivity and its applications*, Kluwer Pub., Dordrecht, 1996.
- A. Pinkus: *Totally positive matrices.* Cambridge Tracts in Mathematics, **181**. Cambridge University Press, Cambridge, 2010.
- S. Fallat, C.R. Johnson: Totally nonnegative matrices. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, New Jersey, 2011.

Surveys on Total Positivity

- T. Ando: Totally positive matrices. *Linear Algebra Appl.* **90** (1987), 165-219.
- S. Fomin, A. Zelevinsky: Total positivity: Tests and Parameterizations. *Math. Intelligencer* **22** (2000), 23-33.
- S. Fallat: Bidiagonal factorizations of totally nonnegative matrices. *Amer. Math. Monthly* **108** (2001), 697-712.
- G. Lusztig: A survey of total positivity. *Milan J. Math.* **76** (2008), 125-134.
- J.M. P.: Tests for the recognition of total positivity. *SeMA Journal* **62** (2013), 61-73.

Conferences, Symposia, and Workshops on Total Positivity

- **1994** International Workshop on Total positivity and its Applications. Jaca. Org.: M. Gasca and C.A. Micchelli.

- **1997** Symposium “Total Positivity and CAGD” in The fourth International Conference on Mathematical Methods for Curves and Surfaces. Lillehammer (Noruega). Org.: J.M. Carnicer.
- **1999** Symposium “Total Positivity” in The eighth Conference of the International Linear Algebra Society (ILAS). Barcelona. Org.: T. Ando and J. Garloff.
- **2009** Invited Symposium “Total Positivity” in the conference Positive Systems: Theory and Applications POSTA09. Valencia. Org.: J. Garloff and J.M. P.
- **2011** Invited Symposium “Total Positivity: recent advances in theory and applications” in The XVII Conference of the International Linear Algebra Society (ILAS). Braunchsweig (Alemania). Org.: P. Koev and J.M. P.
- **2013** Symposium “Matrices and Total Positivity” in The XVIII Conference of the International Linear Algebra Society (ILAS). Providence (USA). Org.: J. Delgado, S. Fallat and J.M. P.

Applications of these matrices:

- **Statistics**

  S. Karlin: *Total Positivity*, Vol. I, Stanford U.P., Calif., 1968.

  L.D. Brown, I.M. Johnstone, K.B. MacGibbon: Variation diminishing transformations: a direct approach to total positivity and its statistical applications, *J. Amer. Statistical Assoc.* **76** (1981), 824-832.

- **Mechanical systems**

  F.R. Gantmacher, M.G. Krein: *Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme.* Akademie-Verlag, Berlin, 1960. Revised English version: *Oscillation matrices*, AMS, Providence, 2002.

- **Real and complex analysis**

I.I. Hirshman, D.V. Widder: *The convolution transform*, Princeton University Press, Princeton, New Jersey, 1955.

I.P. Kamynin, I.V. Ostrowskii: Zero sets of entire Hermitian-polsitive functions. *Siberian Math. J.* **23** (1982), 344-357.

- **Mathematical Biology**

S. Karlin, U. Liberman: Theoretical recombination processes incorporating interference effects. *Theor. Popul. Biol.* **46** (1994), 198-231.

- **Approximation Theory**

C. de Boor, R. De Vore: A geometric proof of total positivity for spline interpolation. *Math. Comput.* **172** (1985), 497-504.

M. Gasca, C.A. Micchelli (Eds.): *Total positivity and its applications*, Kluwer Pub., Dordrecht, 1996.

- **Computer Aided Geometric Design**

J.M. P. (Ed:): *Shape preserving representations in Computer-Aided Geometric Design*, Nova Science Publishers, Commack (New York), 1999.

- **Combinatorics**

F. Brenti: Combinatorics and Total Positivity. *J. Combin. Theory Ser. A* **71** (1995), 175-218.

J.M. P.: On the relationship between graphs and totally positive matrices. *SIAM J. Matrix Anal. Appl.* **19** (1998), 369-377.

- **Stochastic processes**

S. Karlin, J. McGregor, A characterization of birth and death processes, *Proc. Nat. Acad. Sci. U.S.A.* 45:375-379 (1959).

S. Karlin and J. McGregor, Classical diffusion processes and total positivity. *J. Math. Anal. Appl.* 1:163-183 (1960).

- **Economy**

E. Salinelli, C. Sgarra. Correlation matrices of yields and total positivity. *Linear Algebra Appl.* **418** (2006), 682-692.

- **Quantum groups**

S. Fomin, A. Zelevinsky: Double Bruhat cells and total positivity. *J. Amer. Math. Soc.* **12** (1999), 335-380.

S. Fomin, A. Zelevinsky: Total positivity: Tests and Parameterizations. *Math. Intelligencer* **22** (2000), 23-33.

Lusztig, George Introduction to quantum groups. Modern Birkhuser Classics. Birkhuser/Springer, New York, 2010.

Basic properties

$n \times m$ matrix A, $Q_{k,n}$, $k \leq \min\{n, m\}$, $\alpha \in Q_{k,n}$, $\beta \in Q_{k,m}$, submatrix $A[\alpha|\beta]$, principal submatrix $A[\alpha] := A[\alpha|\alpha]$

Cauchy-Binet formula:

$$\det(AB)[\alpha|\beta] = \sum_{\omega \in Q_{k,n}} \det A[\alpha|\omega] \det B[\omega|\beta], \ \textit{with}\ \alpha, \beta \in Q_{k,n}.$$

Corollary. The product of TP matrices is a TP matrix

Shadows' lemma:

If $A = (a_{ij})_{1\le i,j\le n}$ is TP and $a_{ij} = 0$ for some i, j, then one of the following properties hold:

(i) $A[i|1,\dots,n] = 0$,

(ii) $A[1,\dots,n|j] = 0$,

(iii) $a_{rs} = 0$, for all $s \le j$, $i \le r$,

(iv) $a_{rs} = 0$, for all $r \le i$, $j \le s$.

Proof . Since (ii) does not hold, $a_{tj} \ne 0$ for some $t \ne i$. Assume now that $t > i$. Then $a_{il} = 0$ for all $l > j$ because otherwise $\det A[i,t|j,l] = -a_{tj}a_{il} < 0$, which is a contradiction. Since (i) does not hold then $a_{ih} \ne 0$ for some $h < j$. Then $a_{rs} = 0$ for all $r < i, s \ge j$ because otherwise $\det A[r,i|h,s] = -a_{ih}a_{rs} < 0$, which is a contradiction and we have proved that (iv) holds. Analogously it can be proved that if $t < i$ then (iii) holds. ■

Theorem. If $A = (a_{ij})_{1 \le i,j \le n}$ is a **nonsingular TP matrix**, then $\det[\alpha] > 0$ for all $\alpha \in Q_{k,n}$ and $k \le n$. In particular, $a_{ii} > 0$ for all $i = 1, \dots, n$.

Staircase structure of nonsingular TP matrices:

Corollary. A nonsingular TP matrix $A = (a_{ij})_{1 \le i,j \le n}$ satisfies the following conditions

(i) $a_{11} \neq 0,\ a_{22} \neq 0,\ \dots, a_{nn} \neq 0$;
(ii) $a_{ij} = 0$, $i > j \Rightarrow a_{kl} = 0$, for all $s \le j$, $i \le k$;
(iii) $a_{ij} = 0$, $i < j \Rightarrow a_{rs} = 0$, for all$r \le i$, $j \le s$.

Inverses

$$J := \mathrm{diag}(1, -1, 1, \dots, (-1)^{n-1})$$

Theorem. $A = (a_{ij})_{1 \le i,j \le n}$ is an SR matrix of signature $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ if and only if $JA^{-1}J$ is SR of signature $\varepsilon = (\varepsilon_{n-1}\varepsilon_n, \dots, \varepsilon_{n-i}\varepsilon_n, \dots, \varepsilon_1\varepsilon_n, \varepsilon_n)$.

Corollary. A nonsingular matrix A is TP if and only if $JA^{-1}J$ is TP.

A nonsingular matrix A with positive diagonal elements and nonpositive off-diagonal elements is an M-matrix if $A^{-1} \ge 0$.

Theorem. Let A be a nonsingular M-matrix. Then A^{-1} is TP if and only if A is tridiagonal.

Spectral properties

Proposition. Let A be a TP matrix. Then all the eigenvalues of A are nonnegative.

Proposition. Let A be an STP matrix. Then all the eigenvalues of A are positive and simple.

O.D. Kellogg: Orthogonal Function Sets Arising from Integral Equations. *Amer. J. Math.* **40** (1918), 145-154.

Many nice properties of eigenvalues and eigenvectors of TP, STP, SR and SSR matrices.

For instance, If A is an $n \times n$ STP matrix with eigenvalues

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n (> 0)$$

and $u_1, \ldots, u_n$ are their associated eigenvectors, then the number of changes of strict sign in the ordered sequence of components of u_k is $k-1$.

Characterizations through a reduced number of minors

For SSR and STP matrices:

- Definition: **all** minors ($\binom{2n}{n} - 1$ minors).
- all minors with **consecutive** rows and columns ($n(n+1)(2n+1)/6$ minors).

M. Fekete, G. Polya: Uber ein Problem von Laguerre. *Rend. C. M. Palermo* **34** (1912), 89-120.

For STP matrices:

- all minors with **initial** rows and consecutive columns and **initial** columns and consecutive rows ($n(n-1)$ menores).

M. Gasca, J.M. P.: Total positivity and Neville elimination. *Linear Algebra Appl.* **165** (1992), 25-44.

Factorizations in terms of bidiagonal matrices

If K is TP and nonsingular, then we can write

$$K = L_{n-1} L_{n-2} \cdots L_1 D U_1 \cdots U_{n-2} U_{n-1},$$

where the matrices L_i (resp., U_i) are nonnegative lower (resp., upper) triangular **bidiagonal** with unit diagonal and D is a **diagonal** matrix with **positive** diagonals.

Uniqueness of the factorization in:

M. Gasca, J.M. P.: A matricial description of Neville elimination with applications to total positivity. *Linear Alg. Appl.* **202** (1994), 33–54.

If K is SR and nonsingular, then we can write

$$K = L_{n-1} L_{n-2} \cdots L_1 D U_1 \cdots U_{n-2} U_{n-1},$$

where the matrices L_i (resp., U_i) are nonnegative lower (resp., upper) triangular **bidiagonal** with unit diagonal and D is a **diagonal** matrix with nonzero diagonals:

M. Gasca, J.M. P.: A test for strict sign-regularity. *Linear Alg. Appl.* **197-198** (1994), 133–142.

If K is nonsingular TP and stochastic, then we can write

$$K = F_{n-1}F_{n-2}\cdots F_1 G_1 \cdots G_{n-2}G_{n-1},$$

with

$$F_i = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & \alpha_{i+1,1} & 1-\alpha_{i+1,1} & & \\ & & & & \ddots & \ddots & \\ & & & & & \alpha_{n,n-i} & 1-\alpha_{n,n-i} \end{pmatrix}$$

and

$$G_i = \begin{pmatrix} 1 & 0 & & & & & \\ & \ddots & \ddots & & & & \\ & & 1 & 0 & & & \\ & & & 1-\alpha_{1,i+1} & \alpha_{1,i+1} & & \\ & & & & \ddots & \ddots & \\ & & & & & 1-\alpha_{n-i,n} & \alpha_{n-i,n} \\ & & & & & & 1 \end{pmatrix},$$

where, $\forall\,(i,j)$, $0 \leq \alpha_{i,j} < 1$.

Interpretation in **CAGD** of this factorization as a **corner cutting algorithm**: the most important algorithms in CAGD.

The bidiagonal factorization of nonsingular TP matrices is associated to an elimination procedure alternative to Gauss elimination called **Neville elimination**. It requires $O(n^3)$ elementary operations to check if an $n \times n$ matrix is either TP or STP:

M. Gasca, J.M. P.: Total positivity and Neville elimination. *Linear Algebra Appl.* **165** (1992), 25-44.

Neville elimination produces zeros in each column by adding to each row an adequate multiple of the previous one (instead of a multiple of the pivot row as in Gauss elimination).

Effects of finite precision arithmetic on numerical algorithms:

- Roundoff errors.
- Data uncertainty.

Key concepts:

- *Conditioning*: it measures the sensibility of solutions to perturbations of data.
- *Growth factor*: it measures the relative size of the intermediate computed numbers with respect to the initial coefficients or to the final solution.
- *Backward error*: if the computed solution is the exact solution of a perturbated problem, it measures such perturbation.
- *Forward error*: it measures the distance between the exact solution and the computed solution.

(Forward error) $\leq$ *(Backward error)* $\times$ *(Condition)*

Growth factor

$$\rho_n^W(A) := \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{\max_{i,j} |a_{ij}|}$$

ρ_n^W associated with partial pivoting of an $n \times n$ matrix is bounded above by 2^n. ρ_n^W associated with complete pivoting of an $n \times n$ matrix is "usually" bounded above by n.

Gauss elimination of a symmetric positive definite matrix (without row or column exchanges) presents $\rho_n^W = 1$.

Amodio and Mazzia have introduced the growth factor

$$\rho_n(A) := \frac{\max_k \|A^{(k)}\|_\infty}{\|A\|_\infty}.$$

P. Amodio, F. Mazzia: A new approach to backward error analysis of LU factorization, BIT 39 (1999) pp. 385–402.

Condition number

$$\kappa(A) := \|A\|_\infty \, \|A^{-1}\|_\infty .$$

The Skeel condition number:

$$\mathrm{Cond}(A) := \|\, |A^{-1}|\, |A| \,\|_\infty .$$

- $\mathrm{Cond}(A) \le \kappa(A)$
- $\mathrm{Cond}(DA) = \mathrm{Cond}(A)$ for any nonsingular diagonal matrix D

Accurate calculation: the relative error is bounded by $\mathcal{O}(\varepsilon)$, where ε is the machine precision.

Admissible operations in algorithms with high relative precision: products, quotients, sums of numbers of the same sign and sums/subtractions of exact data:

The only **forbidden** operation is true subtraction, due to possible cancellation in leading digits.

In order to guarantee accurate computations for some special classes of matrices, it is crucial to find an **adequate parametrization** of the special classes of matrices:

- For nonsingular totally positive matrices: the multipliers of its Neville elimination.

Neville elimination produces zeros in each column by adding to each row an adequate multiple of the previous one (instead of a multiple of the pivot row as in Gauss elimination).

The bidiagonal factorization of nonsingular TP matrices is associated to an elimination procedure alternative to Gauss elimination called **Neville elimination**. It requires $O(n^3)$ elementary operations to check if an $n \times n$ matrix is either TP or STP:

M. Gasca, J.M. P.: Total positivity and Neville elimination. *Linear Algebra Appl.* **165** (1992), 25-44.

- **A stable test to check if a matrix is STP or TP.**

Neville elimination (NE)

If A is a nonsingular matrix of order n, it consists of $n-1$ steps:

$$A = A^{(1)} \to A^{(2)} \to \ldots \to A^{(n)} = U,$$

$$A^{(t)} = \begin{pmatrix} a_{11}^{(t)} & a_{12}^{(t)} & \cdots & \cdots & \cdots & \cdots & a_{1n}^{(t)} \\ 0 & a_{22}^{(t)} & \cdots & \cdots & \cdots & \cdots & a_{2n}^{(t)} \\ \vdots & 0 & \ddots & & & & \vdots \\ \vdots & \vdots & & \ddots & & & \vdots \\ \vdots & \vdots & & & a_{tt}^{(t)} & \cdots & a_{tn}^{(t)} \\ \vdots & \vdots & & & \vdots & & \vdots \\ 0 & 0 & \cdots & \cdots & a_{nt}^{(t)} & \cdots & a_{nn}^{(t)} \end{pmatrix}.$$

$$a_{ij}^{(t+1)} = \begin{cases} a_{ij}^{(t)} & i \leq t \\ a_{ij}^{(t)} - \frac{a_{it}^{(t)}}{a_{i-1,t}^{(t)}} a_{i-1,j}^{(t)} & i \geq t+1,\ a_{i-1,t}^{(t)} \neq 0 \\ a_{ij}^{(t)} & i \geq t+1,\ a_{i-1,t}^{(t)} = 0 \end{cases}$$

The element

$$p_{ij} := a_{ij}^{(j)}, \quad 1 \leq j \leq i \leq n,$$

is called (i, j) *pivot* of the NE of A.

The *complete Neville elimination* (CNE) of A: NE of A until obtaining U and NE of $V := U^T$.

TESTS

Growth factor:

$$\rho := \max\{\frac{\max_{i,j,k}|a_{ij}^{(k)}|}{\max_{i,j}|a_{ij}|}, \frac{\max_{i,j,k}|v_{ij}^{(k)}|}{\max_{i,j}|a_{ij}|}\}(\geq 1)$$

Theorem.

(i) *A matrix* $A = (a_{ij})_{1\leq i\leq n}^{1\leq j\leq m}$ *is STP if and only if we can perform the CNE of A without row or column exchanges and, for all* $k = 1, \ldots, r = \min\{n, m\}$, $a_{ij}^{(k)} > 0$ *for all* $i, j \geq k$ *and* $v_{ij}^{(k)} > 0$ *for all* i, j *with* $i \geq j \geq k$.

(ii) *The test suggested by (i) to check if A is STP can be performed in* $\mathcal{O}(s^3)$ $(s = \max\{n, m\})$ *elementary operations.*

(iii) *The growth factor is* $\rho = 1$.

J.M. P.: Tests for the recognition of total positivity. *SeMA Journal* **62** (2013), 61-73.

Theorem.

(i) A square matrix $A = (a_{ij})_{1\leq i,j\leq n}$ is nonsingular and TP if and only if we can perform the CNE of A without row or column exchanges, the diagonal pivots are positive and, for all $k = 1, \ldots, n$, $a_{ij}^{(k)} \geq 0$ for all $i, j \geq k$ and $v_{ij}^{(k)} > 0$ for all i, j with $i \geq j \geq k$.

(ii) The test suggested by (i) to check the total positivity of A can be performed in $\mathcal{O}(n^3)$ elementary operations.

(iii) The growth factor is $\rho = 1$.

Definition. A nonsingular TP matrix such that a minor is nonzero if and only if its diagonal entries are nonzero is called almost strictly totally positive (ASTP).

A Hurwitz matrix with positive leading principal minors is ASTP and a nonsingular B-spline collocation matrix is ASTP.

M. Gasca, C.A. Micchelli, J.M. P.: Almost strictly totally positive matrices, *Numerical Algorithms* **2** (1992), pp. 225-236.

J.M. Carnicer, J.M. P.: Spaces with almost strictly totally positive bases, *Mathematische Nachrichten* **169** (1994), pp. 69-79.

D. Dimitrov, J.M. P.: Almost strict total positivity and a class of Hurwitz polynomials, *J. Approx. Theory* **132** (2005), pp. 212-223.

M. Gasca, J.M. P.: Characterizations and decompositions of almost strictly positive matrices, *SIAM J. Matrix Anal. Appl.* **28** (2006), pp. 1-8.

J.M. P.: An optimal test for almost strict total positivity, *Linear Algebra and its Applications* **448** (2014), pp. 274-284.

Neville elimination leads to a factorization of a nonsingular totally positive matrix in terms of **bidiagonal** factors, and the elements appearing in the factorization are natural parameters of the matrix.

This factorization has been used to obtain **accurate computations** with nonsingular totally positive matrices. In particular, accurate computation of their SVDs and eigenvalues.

P. Koev: Accurate Eigenvalues and SVDs of Totally Nonnegative Matrices, *SIAM J. Matrix Anal. Appl.* **27** (2005), 1-23.

J. Demmel and P. Koev: The Accurate and Efficient Solution of a Totally Positive Generalized Vandermonde Linear System, *SIAM J. Matrix Anal. Appl.* **27** (2005), 142-152.

P. Koev: Accurate computations with totally nonnegative matrices, *SIAM J. Matrix Anal. Appl.* **29** (2007), no. 3, 731–751.

- **A stable test for strict sign-regularity.**

V. Cortes, J.M. P.: Sign regular matrices and Neville elimination, *Linear Algebra Appl.* **421** (2007), 53-62.

V. Cortes, J.M. P.: Decompositions of strictly sign regular matrices. *Linear Algebra and its Applications* **429** (2008), 1071-1081.

Neville elimination with a pivoting strategy called **two-determinant pivoting**.

An efficient test to check if a given matrix is SSR in:
V. Cortes, J.M. P.: A stable test for strict sign regularity. *Math. Comp.* **77** (2008), 2155-2171.
The test has **optimal growth factor**.

M. García-Esnaola and J.M. P., *Sign consistent linear programming problems.* Optimization 58 (2009), 935–946.

Continuous extensions of TP and STP matrices

Let $X, Y \subseteq \mathbf{R}$. The real function (or *kernel*) $K(x,y)$ is *TP* (resp., STP) if

$$(K(x_i, y_j))_{1\le i\le n}^{1\le j\le n}$$

is TP (resp., STP) for every choice of $x_1 < \cdots < x_n$ in X, $y_1 < \cdots < y_m$ in Y and all possible n, m

$(u_0, \ldots, u_n)$ basis of a vector space of functions.

Collocation matrices: for $a \le t_0 < t_1 < \cdots < t_n \le b$

$$M\begin{pmatrix} u_0, \ldots, u_n \\ t_0, \ldots, t_n \end{pmatrix} := (u_j(t_i))_{i,j=0,\ldots,n}$$

Definition. A system of functions $(u_0, \ldots, u_n)$ is *TP* (resp., STP) if all its collocation matrices are TP (resp., STP).

An application to CAGD

Given a sequence of functions $u_0, \ldots, u_n$ on $[a, b]$ such that

$$\sum_{i=0}^{n} u_i(t) = 1 \quad \forall\, t \in [a, b]$$

(i.e., $(u_0, \ldots, u_n)$ is normalized) and a sequence of points in $\mathbf{R}^k$ $(C_0, \ldots, C_n)$, we may define a curve

$$\gamma(t) = \sum_{i=0}^{n} C_i u_i(t), \quad t \in [a, b].$$

The points C_i, $i = 0, \ldots, n$ are the *control points*.

The polygon $C_0 \cdots C_n$ is called the *control polygon* of the curve γ.

Definition. A normalized system of nonnegative functions $(u_0, \ldots, u_n)$ is called *blending*.

A system $(u_0, \ldots, u_n)$ is blending if and only if the generated curves always lie in the convex hull of the control polygon (*convex hull property*).

It is desirable that the curve “mimics” the control polygon and that the control polygon even “exaggerates” the shape of the curve. TP systems of functions satisfy it due to the *variation diminishing* properties of totally positive matrices.

Definition. A TP basis $(u_0, \ldots, u_n)$ is *normalized totally positive* (NTP) if

$$\sum_{i=0}^{n} u_i(t) = 1, \quad \forall t \in I.$$

Endpoint interpolation property: the first control point always coincides with the start point of the curve and the last control point always coincides with the final point of the curve.

Theorem. *If a given basis satisfies simultaneously the variation diminishing, the endpoint interpolation and the convex hull properties then it is NTP.*

Definition. *A TP basis* $(b_0, \ldots, b_n)$ *of a space of functions* $\mathcal{U}$ *such that, for any TP basis* $(v_0, \ldots, v_n)$ *of* $\mathcal{U}$ *there exists a TP matrix* K *satisfying*

$$(v_0, \ldots, v_n) = (b_0, \ldots, b_n)K$$

is called a B-basis of $\mathcal{U}$.

Theorem. *Let* $\mathcal{U}$ *be a vector space of functions which has a totally positive basis. Then:*

(i) *There exists a B-basis* $\mathbf{b} = (b_0, \ldots, b_n)^T$ *of* $\mathcal{U}$.

(ii) *A basis of* $\mathcal{U}$ *is a B-basis of* $\mathcal{U}$ *if and only if it is of the form* $(d_0 b_0, \ldots, d_n b_n)$, *where* $d_k > 0$ *for all* k.

(iii) *A basis* $\mathbf{u}$ *of* $\mathcal{U}$ *is TP if and only if* $\mathbf{u}^T = \mathbf{b}^T K$ *and* K *is a nonsingular TP matrix.*

J.M. Carnicer and J.M. P., "Totally positive bases for shape preserving curve design and optimality of B-splines", *Computer Aided Geometric Design* **11** (1994), 633-654.

Theorem. *Let $\mathcal{U}$ be a vector space of functions which has a NTP basis. Then:*

(i) *There exists a unique normalized B-basis* $\mathbf{b}$ *of* $\mathcal{U}$.

(ii) *A basis* $\mathbf{u}$ *of* $\mathcal{U}$ *is NTP if and only if* $\mathbf{u}^T = \mathbf{b}^T K$ *and* K *is a nonsingular stochastic TP matrix.*

A normalized B-basis has optimal shape preserving and stability properties.

Theorem. *A basis is a normalized B-basis if and only if it satisfies the least variation diminishing, the endpoint interpolation and the convex hull properties simultaneously.*

Book on the subject: "Shape preserving representations in Computer-Aided Geometric Design" (J.M. P., editor). Nova Science Publishers, Commack (New York), 1999.

If a space does not possess normalized B-basis, then it does not possess any shape preserving representation

Optimal shape preserving properties of the Bernstein basis:

J.M. Carnicer and J.M. P., "Shape preserving representations and optimality of the Bernstein basis" (1993). *Advances in Computational Mathematics* **1**, pp. 173-196.

The Bernstein basis is a normalized B-basis of the space of polynomials of degree less than or equal to n on a compact interval $[a, b]$:

$$b_i(t) := \binom{n}{i} \left(\frac{t-a}{b-a}\right)^i \left(\frac{b-t}{b-a}\right)^{n-i}, \quad i = 0, \dots, n.$$

The collocation matrices of the Bernstein basis are called **Bernstein-Vandermonde matrices**.

Optimal stability properties of the Bernstein basis.

Fastest progressive iterative approximation and optimal conditioning of the collocation matrices

Theorem. The progressive iterative approximation process converges for any nonsingular collocation matrix B of an NTP basis.

H. LIN, H. BAO, G. WANG (2005), Totally positive bases and progressive iteration approximation, Computer & Mathematics with Applications 50, 575-586.

Theorem. Given a space U with an NTP basis, the normalized B-basis of U provides a progressive iterative approximation with the fastest convergence rates among all NTP bases of U.

DELGADO J., P. J.M.: "Progressive iterative approximation and bases with the fastest convergence rates" (2007). *Computer Aided Geometric Design* **24**, pp. 10-18.

Optimal conditioning of Bernstein-Vandermonde matrices:

DELGADO J., P. J.M.: "Optimal conditioning of Bernstein collocation matrices" (2009). *SIAM J. Matrix Anal. Appl.* **31**, 990-996.

Theorem. Let $(b_0, \ldots, b_n)$ be the **Bernstein basis**, let $(v_0, \ldots, v_n)$ be another NTP basis of P_n on $[0, 1]$, let $0 \leq t_0 < t_1 < \cdots < t_n \leq 1$ and $V := M\begin{pmatrix} v_0,\ldots,v_n \\ t_0,\ldots,t_n \end{pmatrix}$ and $B := M\begin{pmatrix} b_0,\ldots,b_n \\ t_0,\ldots,t_n \end{pmatrix}$. Then

$$\kappa_\infty(B) \leq \kappa_\infty(V).$$

$$\text{Cond}(A) := \| \, |A^{-1}| \, |A| \, \|_\infty.$$

Theorem. Let $(b_0, \ldots, b_n)$ be the **Bernstein basis**, let $(v_0, \ldots, v_n)$ be another TP basis of P_n on $[0, 1]$, let $0 \le t_0 < t_1 < \cdots < t_n \le 1$ and $V := M\begin{pmatrix} v_0, \ldots, v_n \\ t_0, \ldots, t_n \end{pmatrix}$ and $B := M\begin{pmatrix} b_0, \ldots, b_n \\ t_0, \ldots, t_n \end{pmatrix}$. Then

$$\text{Cond}(B^T) \le \text{Cond}(V^T).$$

Theorem. The **minimal eigenvalue** of a **normalized B-basis** collocation matrix is always **greater than or equal to** the minimal eigenvalue of the corresponding collocation matrix of **another NTP** basis of the space.

Key facts of the proof:

- U has a unique normalized B-basis $(b_0, \ldots, b_n)$. If $(v_0, \ldots, v_n)$ is another NTP basis of U, then there exists a stochastic TP matrix K such that

$$(v_0, \ldots, v_n) = (b_0, \ldots, b_n)K. \tag{1}$$

- Let $t_0 < t_1 < \cdots < t_n$ in the domain of the functions of U such that $V := M\begin{pmatrix} v_0,\ldots,v_n \\ t_0,\ldots,t_n \end{pmatrix}$ and $B := M\begin{pmatrix} b_0,\ldots,b_n \\ t_0,\ldots,t_n \end{pmatrix}$ are nonsingular. By (1), $V = BK$.
- Let us consider the $(n+1)\times(n+1)$ matrix $J := \mathrm{diag}(1, -1, 1, \ldots, (-1)^n)$. Since $J^{-1} = J$, $JV^{-1}J$ is similar to V^{-1} and $JB^{-1}J$ is similar to B^{-1}, it is sufficient to prove that

$$\rho(JV^{-1}J) \geq \rho(JB^{-1}J). \tag{2}$$

- (Wienlandt's theorem) Let M be a nonnegative matrix with maximal eigenvalue r, and let C be a complex matrix dominated by M, then $r = \rho(M) \geq \rho(C)$.
- The matrices $JV^{-1}J$ and $JB^{-1}J$ are also TP and so nonnegative. Then, since $JV^{-1}J = J(BK)^{-1}J = (JK^{-1}J)(JB^{-1}J)$, in order to prove the theorem it is sufficient to see that

$$JB^{-1}J \leq JV^{-1}J = (JK^{-1}J)(JB^{-1}J). \tag{3}$$

- Since K is TP and stochastic, we can write

$$K = F_{n-1}F_{n-2}\cdots F_1G_1\cdots G_{n-2}G_{n-1}, \tag{4}$$

with

$$F_i = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & \alpha_{i+1,1} & 1-\alpha_{i+1,1} & & \\ & & & & \ddots & \ddots & \\ & & & & & \alpha_{n,n-i} & 1-\alpha_{n,n-i} \end{pmatrix}$$

and

$$G_i = \begin{pmatrix} 1 & 0 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 1 & 0 & & & & \\ & & & 1-\alpha_{1,i+1} & \alpha_{1,i+1} & & & \\ & & & & \ddots & \ddots & & \\ & & & & & 1-\alpha_{n-i,n} & \alpha_{n-i,n} & \\ & & & & & & 1 & \end{pmatrix},$$

where, $\forall\,(i,j)$, $0 \leq \alpha_{i,j} < 1$.

- If we denote by $U(\lambda_i)$ (resp., $L(\lambda_i)$) the *bidiagonal*, nonsingular and upper (resp., lower) triangular matrix with at most one nonzero off-diagonal element

$$U(\lambda_i) = \begin{pmatrix} 1 & 0 & & & & & & \\ & 1 & 0 & & & & & \\ & & \ddots & \ddots & & & & \\ & & & 1-\lambda_i & \lambda_i & & & \\ & & & & \ddots & \ddots & & \\ & & & & & 1 & 0 \\ & & & & & & 1 \end{pmatrix}, \quad 0 \leq \lambda_i < 1 \qquad (5)$$

(resp.,

$$L(\lambda_i) = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & \lambda_i & 1-\lambda_i & & & \\ & & & \ddots & \ddots & & \\ & & & & & 1 & \\ & & & & & 0 & 1 \end{pmatrix}, \quad 0 \leq \lambda_i < 1), \tag{6}$$

then we can write

$$F_i = L(\alpha_{i+1,1}) \cdots L(\alpha_{n,n-i}) \tag{7}$$

and

$$G_i = U(\alpha_{n-i,n}) \cdots U(\alpha_{1,i+1}). \tag{8}$$

- From (4), (7) and (8), we conclude that $K = \Pi_{i=1}^{r} E_i$, where r is a positive integer and E_i is a matrix of the form (5) or (6). Therefore

$$JK^{-1}J = \Pi_{i=1}^{r}(JE_i^{-1}J).$$

We have seen that $JB^{-1}J$ is nonnegative and we can observe that all matrices $JE_i^{-1}J$ are also nonnegative. So, in order to prove (3), it is sufficient to see that if we multiply a nonnegative matrix A by a matrix of the form $JE_i^{-1}J$, we obtain a new matrix whose entries are greater than or equal to the entries of A. But this last property is an immediate consequence of the form of each $JE_i^{-1}J$:

$$\begin{pmatrix} 1 & 0 & & & & & & \\ & 1 & 0 & & & & & \\ & & \ddots & \ddots & & & & \\ & & & \frac{1}{1-\lambda_i} & \frac{\lambda_i}{1-\lambda_i} & & & \\ & & & & \ddots & \ddots & & \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \frac{\lambda_i}{1-\lambda_i} & \frac{1}{1-\lambda_i} & & & & \\ & & & \ddots & \ddots & & & \\ & & & & & 1 & \\ & & & & & 0 & 1 \end{pmatrix},$$

with $0 \leq \lambda_i < 1$, and of the fact that $1/(1-\lambda_i) \geq 1$.

A more general class of matrices with bidiagonal decomposition and accurate computations

$$
L^{(k)} = \begin{pmatrix}
1 & & & & & & \\
0 & 1 & & & & & \\
 & \ddots & \ddots & & & & \\
 & & 0 & 1 & & & \\
 & & & l_{n-k}^{(k)} & 1 & & \\
 & & & & \ddots & \ddots & \\
 & & & & & l_{n-1}^{(k)} & 1
\end{pmatrix}
$$

$$U^{(k)} = \begin{pmatrix} 1 & 0 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 1 & 0 & & & & \\ & & & 1 & u_{n-k}^{(k)} & & & \\ & & & & \ddots & \ddots & & \\ & & & & & 1 & u_{n-1}^{(k)} \\ & & & & & & 1 \end{pmatrix}$$

Let A be a nonsingular $n \times n$ matrix. Suppose that we can write A as a product of bidiagonal matrices

$$A = L^{(1)} \cdots L^{(n-1)} D U^{(n-1)} \cdots U^{(1)}, \tag{1}$$

with

1. $d_i \neq 0$ for all i,
2. $l_i^{(k)} = u_i^{(k)} = 0$ for $i < n-k$,
3. $l_i^{(k)} = 0 \Rightarrow l_{i+s}^{(k-s)} = 0$ for $s = 1, \ldots, k-1$ and $u_i^{(k)} = 0 \Rightarrow u_{i+s}^{(k-s)} = 0$ for $s = 1, \ldots, k-1$.

Then we denote (1) by $\mathcal{BD}(\mathcal{A})$, a bidiagonal decomposition of A satisfying the conditions of this definition.

Theorem. Let A be a nonsingular matrix. If a $\mathcal{BD}(\mathcal{A})$ exists, then it is unique.

Let us denote by ε the vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)$ with $\varepsilon_j \in \{\pm 1\}$ for $j = 1, \ldots, m$, which will be called a *signature*.

Given a signature $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-1})$ and a nonsingular $n \times n$ matrix A, we say that A has a **signed bidiagonal decomposition with signature ε** if there exists a $\mathcal{BD}(\mathcal{A})$ such that

$d_i > 0$ for all i,

$l_i^{(k)} \varepsilon_i \geq 0$, $u_i^{(k)} \varepsilon_i \geq 0$ for $1 \leq k \leq n-1$ and $n-k \leq i \leq n-1$.

Totally positive matrices and their inverses are particular examples.

BARRERAS A., P. J.M.: "Accurate computations of matrices with bidiagonal decomposition using methods for totally positive matrices" (2013). *Numerical Linear Algebra with Applications* **20**, pp. 413-424.