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Backward Error and Conditioning for the Polynomial Eigenvalue Problem

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2nd ALAMA Courses on Matrix Polynomials

English Electric Company

Analogue computer room at the English Electric Company aerostructures group in August 1955.

Peter Lancaster is in the driver's seat of the life-size cockpit.

Notation

- Consider $P(\lambda) = \sum_{i=0}^{\ell} \lambda^i A_i, \quad A_i \in \mathbb{C}^{n\times n}, \quad A_\ell \neq 0.$
  - For $\ell = 2$ (quadratic) replace P by Q,
  - For $\ell = 1$ (linear) replace P by L.
- Assume $P(\lambda)$ is **regular**, i.e., $\det(P(\lambda)) \not\equiv 0$.
- PEP: Find scalars λ and nonzero $x, y \in \mathbb{C}^n$ satisfying $P(\lambda)x = 0$ and $y^* P(\lambda) = 0$.
  - λ is an e'val,
  - x, y are corresponding right and left e'vecs.

Conditioning and Backward Error

- **Condition number** measures sensitivity of the solution of a problem to perturbations in the data.
  - Conditioning is a property of the problem.
- **Backward error** measures how far a problem has to be perturbed for an approximate solution to be an exact solution of the perturbed problem.
  - Backward error characterizes the stability of a method for solving the problem.

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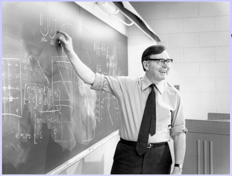
forward error $\lesssim$ **cond. number** $\times$ **backward error**.

Outline

- Eigenvalue condition number for $P(\lambda)$
- Backward error for approximate eigenpairs for $P(\lambda)$
- Sensitivity and stability of linearization process
- Pseudospectra

Wilkinson Condition Number



Simple λ, $Ax = \lambda x$, $y^*A = \lambda y^*$. Let

$$(A + \Delta A)(x + \Delta x) = (\lambda + \Delta\lambda)(x + \Delta x).$$

Expand and drop 2nd order terms:

$$A\Delta x + \Delta A\, x = \lambda \Delta x + \Delta\lambda\, x.$$

Premultiply by y^*:

$$\Delta\lambda = \frac{y^*\Delta A x}{y^* x} \quad \Rightarrow \quad |\Delta\lambda| \le \frac{\|\Delta A\|_2 \|y\|_2 \|x\|_2}{|y^* x|}.$$

Attainable bound, hence (absolute) condition number

$$K(\lambda) = \frac{\|y\|_2 \|x\|_2}{|y^* x|}.$$

Eigenvalue Condition Number $\kappa_P(\lambda)$

Assume λ simple, finite and nonzero, $P(\lambda)x = 0$, $y^*P(\lambda) = 0$.

Normwise (relative) condition number can be defined by

$$\kappa_P(\lambda) = \lim_{\epsilon\to 0}\sup\Big\{ \frac{|\Delta\lambda|}{\epsilon|\lambda|} : \big(P(\lambda+\Delta\lambda) + \Delta P(\lambda+\Delta\lambda)\big)(x+\Delta x) = 0, \\ \|\Delta A_i\|_2 \le \epsilon\omega_i,\ i = 0\colon \ell \Big\},$$

with the requirement $\Delta x \to 0$ as $\epsilon \to 0$.

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$$\left. \|\Delta A_i\|_2 \le \epsilon\omega_i,\ i = 0\colon \ell \right\},$$

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The ω_i are nonnegative weights.

- $\omega_i \equiv 1$ gives an **absolute** measure,
- $\omega_i = \|A_i\|_2$ gives a **relative** measure,
- $\omega_i = 0$ forces $\Delta A_i = 0$ and thus keeps A_i **unperturbed**.

Eigenvalue Condition Number (cont.)

Theorem (T.2000)

The normwise eigenvalue condition number is given by

$$\kappa_P(\lambda) = \alpha \frac{\|y\|_2 \|x\|_2}{|\lambda|\,|y^* P'(\lambda) x|}, \quad \alpha = \sum_{i=0}^{\ell} |\lambda|^i \omega_i.$$

- For $P(\lambda) = \lambda I - A$, $\omega_1 = 0$ and $\omega_0 = 1$, recover (relative) Wilkinson condition number, $\kappa(\lambda) = \dfrac{\|x\|_2 \|y\|_2}{|\lambda||y^* x|}$.

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Problems

- κ_P not defined for $\lambda = 0$.
- κ_P not defined for $\lambda = \infty$.

Homogeneous Form

Rewrite

$$P(\alpha,\beta) = \alpha^\ell A_\ell + \cdots + \alpha\beta^{\ell-1}A_1 + \beta^\ell A_0, \qquad (\lambda \equiv \alpha/\beta).$$

- E'vals are pairs $(\alpha,\beta) \neq (0,0)$ s.t. $\det P(\alpha,\beta) = 0$.
  - $\lambda = 0$ corresponds to $\alpha = 0$,
  - $\lambda = \infty$ corresponds to $\beta = 0$.
- $P(\alpha,\beta)x = 0$ is a **bihomogeneous** equation.
- Natural to consider $x \in \mathbb{P}_{n-1}$ and $(\alpha,\beta) \in \mathbb{P}_1$, where $\mathbb{P}_{k-1} = \mathbb{P}(\mathbb{C}^k)$ is the **set of vector lines in $\mathbb{C}^k$**

  (i.e., $\mathbb{P}_{k-1}$ is the quotient of $\mathbb{C}^k \setminus \{0\}$ for the equivalence relation "*$x \sim y$ if and only if $x = \rho y$ for some $\rho \in \mathbb{C} \setminus \{0\}$*.)"

Homogeneous Form (cont.)

Define **condition number of** (α, β) as the norm of a condition operator $K(\alpha, \beta)$,

$$\kappa_P(\alpha, \beta) = \max_{\|\Delta A\| \leq 1} \frac{\|K(\alpha, \beta)\Delta A\|_2}{\|[\alpha, \beta]\|_2},$$

where

- $K(\alpha, \beta)$ is the differential of map from the $(\ell + 1)$-tuple $A = (A_0, \ldots, A_\ell)$ to (α, β) in projective space,
- $\Delta A \equiv (\Delta A_0, \ldots, \Delta A_\ell)$.
- $\|A\| = \|(A_0, \ldots, A_\ell)\| = \|[\omega_0^{-1} A_0, \ldots, \omega_\ell^{-1} A_\ell]\|_F$ ω-weighted Frobenius norm.

First Order Variation Formula

Extension of result of Dedieu (1997):

Theorem (Higham, Mackey, T. 06)

For simple eigenvalue (α, β) *of* P *with* $\|[\alpha, \beta]\|_2 = 1$ *and suff. small* $(\ell + 1)$*-tuples*

$$\Delta A \equiv (\Delta A_0, \ldots, \Delta A_\ell),$$

$\widetilde{P}(\alpha, \beta) = \sum_{i=0}^{\ell} \alpha^i \beta^{\ell-i}(A_i + \Delta A_i)$ *has a simple eigenvalue*

$$(\tilde{\alpha}, \tilde{\beta}) = (\alpha, \beta) + K(\alpha, \beta)\Delta A + o(\|\Delta A\|).$$

where $[\alpha, \beta][\tilde{\alpha}, \tilde{\beta}]^* = 1$.

Condition Number for Homogeneous Form

Can show that, with suitable normalizations,

$$\left|\theta\big((\alpha,\beta),(\tilde{\alpha},\tilde{\beta})\big)\right| \le \kappa_P(\alpha,\beta)\|\Delta A\| + o(\|\Delta A\|).$$

Take sine of both sides to get bound for **chordal distance** between (α,β) and $(\tilde{\alpha},\tilde{\beta})$ (Stewart & Sun, 1990).

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Theorem (Dedieu, T. 2003)

With $\|A\| := \|[\omega_0^{-1}A_0,\dots,\omega_\ell^{-1}A_\ell]\|_F$ *and* (α,β) *simple,*

$$\kappa_P(\alpha,\beta) = \frac{\left(\sum_{i=0}^{\ell}|\alpha|^{2i}|\beta|^{2(\ell-i)}\omega_i^2\right)^{1/2}\|x\|_2\|y\|_2}{\left|y^*(\bar\beta\mathcal{D}_\alpha P - \bar\alpha\mathcal{D}_\beta P)|_{(\alpha,\beta)}x\right|},$$

where $\mathcal{D}_\alpha \equiv \frac{\partial}{\partial\alpha}$, $\mathcal{D}_\beta \equiv \frac{\partial}{\partial\beta}$.

More about Condition Numbers

- MATLAB functions **polyeig** and **quadeig** return $\kappa_P(\alpha, \beta)$.
- Can define e'vec condition number $\kappa_P(x)$ via condition operator $K(x)$.

$$\kappa_P(\alpha, \beta) = \Big(\sum_{i=0}^{\ell} |\alpha|^{2i} |\beta|^{2(\ell-i)} \omega_i^2 \Big)^{1/2} \| \big(\Pi_{v^\perp} P(A, \alpha, \beta)_{|_{x^\perp}} \big)^{-1} \|$$

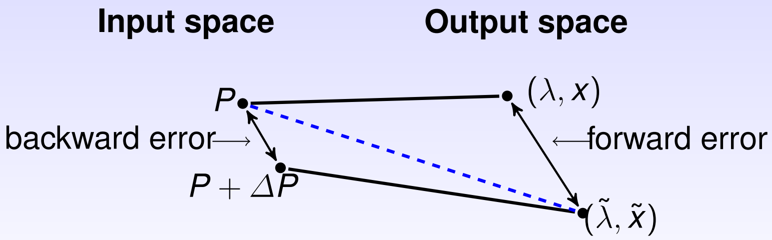
where $v = \bar{\beta} \Delta_\alpha P x - \bar{\alpha} \Delta_\beta P x$, $P(\alpha, \beta)_{|_{x^\perp}}$ is the restriction of $P(\alpha, \beta)$ to $x^\perp$ and $\Pi_{v^\perp}$ is the projection over $v^\perp$.

- Can define structured e'val condition numbers for structured $P(\lambda)$ and get explicit expressions.

Backward Error

Let (λ, x) be a solution of $P(\lambda)x = 0$ and let $(\tilde{x}, \tilde{\lambda})$ be an approximate (computed) solution.

Is $(\tilde{\lambda}, \tilde{x})$ the solution of a slightly perturbed polynomial eigenvalue problem?



Solid line = exact, dotted line = computed.

Backward Error: Definition

Let (λ, x) be an approx. e'pair of $P(\lambda)$ with **finite λ**. Let $\Delta P(\lambda) = \sum_{i=0}^{\ell} \lambda^i \Delta A_i$.

Normwise backward error of (λ, x) can be defined by

$$\eta_P(\lambda, x) = \min\{\, \epsilon : (P(\lambda)+\Delta P(\lambda))x = 0, \|\Delta A_i\|_2 \le \omega_i \epsilon,\ i = 0\colon \ell \,\}.$$

ω_i are nonnegative parameters.

Can show that $$\eta_P(x, \lambda) = \frac{\|P(\lambda)x\|_2}{\left(\sum_{i=0}^{\ell} |\lambda^i|\omega_i\right) \|x\|_2}$$ (T. 2000)

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Problem

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Homogeneous Form

$P(\alpha,\beta)=\sum_{i=0}^{\ell}\alpha^i\beta^{\ell-i}A_i$, $p(|\alpha|,|\beta|)=\sum_{i=0}^{\ell}|\alpha|^i|\beta|^{\ell-i}\omega_i$,

B'err of approx. e'pair:

$$\eta_P(x,\alpha,\beta)=\frac{\|P(\alpha,\beta)x\|_2}{p(|\alpha|,|\beta|)\|x\|_2}.$$

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When e'vecs are not computed use b'err for approx. e'val:

$$\eta_P(\alpha,\beta)=\min_{x\neq 0}\eta_P(x,\alpha,\beta)=\Big(p(|\alpha|,|\beta|)\|[P(\alpha,\beta)]^{-1}\|_2\Big)^{-1},$$

Expressions indep. of choice of α and β.

Standard Solution Process

Find all λ and x satisfying $Q(\lambda)x = (\lambda^2 M + \lambda D + K)x = 0$.

- Commonly solved by **linearization**:
  - **Convert** $Q(\lambda)x = 0$ into $(\mathcal{A} - \lambda\mathcal{B})\xi = 0$, e.g.,

$$\mathcal{A} - \lambda\mathcal{B} = \begin{bmatrix} K & 0 \\ 0 & I \end{bmatrix} - \lambda \begin{bmatrix} -D & -M \\ I & 0 \end{bmatrix}, \quad \xi = \begin{bmatrix} x \\ \lambda x \end{bmatrix}.$$

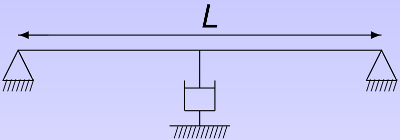
  - **Solve** $(\mathcal{A} - \lambda\mathcal{B})\xi = 0$ with an eigensolver for generalized eigenproblem (e.g., QZ algorithm).
  - **Recover** eigenvectors of $Q(\lambda)$ from those of $\mathcal{A} - \lambda\mathcal{B}$.
- Eigensolver often **absent from numerical libraries**.

Numerical Issues

- **Infinitely many linearizations** to choose from.
- Solving PEPs with a backward stable alg. (e.g., QZ alg.) applied to a linearization can be **backward unstable** for the PEP.
- Linearizations can have **widely varying eigenvalue condition numbers**.

Numerical solution of PEPs requires special attention.

Example 2: Beam Problem



Transverse displacement $u(\mathrm{x}, t)$

$$\rho A \frac{\partial^2 u}{\partial t^2} + c(\mathrm{x}) \frac{\partial u}{\partial t} + EI \frac{\partial^4 u}{\partial \mathrm{x}^4} = 0.$$

$$u(0, t) = u''(0, t) = u(L, t) = u''(L, t) = 0.$$

Finite element method leads to

$$Q(\lambda)v = (\lambda^2 M + \lambda D + K)v = 0$$

with symmetric $M, D, K \in \mathbb{R}^{n \times n}$.

- $M > 0$, $K > 0$, $D \geq 0 \;\Rightarrow$ all e'vals have $\mathrm{Re}(\lambda) \leq 0$.
- D is rank 1. Can show n pure imaginary e'vals.

Eigenvalues of $Q(\lambda) = \lambda^2 M + \lambda D + K$

When M, K are nonsingular then theoretically

$$C_1(\lambda) = \lambda \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} D & K \\ -I & 0 \end{bmatrix},$$

$$L_1(\lambda) = \lambda \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix} + \begin{bmatrix} D & K \\ K & 0 \end{bmatrix}, \quad L_2(\lambda) = \lambda \begin{bmatrix} 0 & M \\ M & D \end{bmatrix} + \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix}$$

have the same eigenvalues as $Q(\lambda)$.

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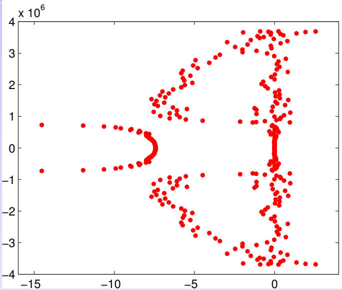
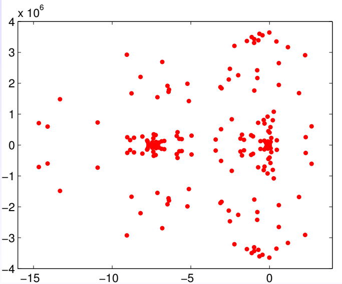
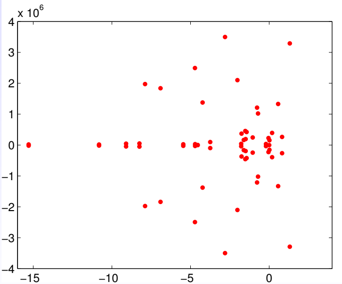
```
coeffs = nlevp('damped_beam',100);
K = coeffs{1}; D = coeffs{2}; M = coeffs{3};
I = eye(2*nele); O = zeros(2*nele);
eval = eig([D K; -I O],-[M O; O I]; % C1
%eval = eig([D K; K O],-[M O; O -K]; % L1
%eval = eig([-M O; O K],-[O M; M D]; % L2
plot(eval,'.r');
```

Computed Spectra of C_1, L_1 and L_2







Sensitivity and Stability of Linearizations

Need tools/framework for analyzing the sensitivity and stability of linearizations $L(\lambda) = \lambda X + Y$ of $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$.

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- **One-sided factorizations** of L typically hold:

$$L(\lambda)F(\lambda) = g \otimes P(\lambda), \qquad g \in \mathbb{C}^{\ell}, \qquad \text{(right sided)}$$
$$E(\lambda)L(\lambda) = h^T \otimes P(\lambda), \qquad h \in \mathbb{C}^{\ell}. \qquad \text{(left sided)}$$

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- For first **companion linearization**, $L(\lambda) = C_1(\lambda)$,

$$C_1(\lambda)(\Lambda \otimes I_n) = e_1 \otimes P(\lambda),$$

and for $\ell = 3$,

$$\begin{bmatrix} \lambda^2 I_n & -(A_0 + \lambda A_1) & -\lambda A_0 \\ \lambda I_n & \lambda A_2 + \lambda^2 A_3 & -A_0 \\ I_n & A_2 + \lambda A_3 & A_1 + \lambda A_2 + \lambda^2 A_3 \end{bmatrix} C_1(\lambda) = I_3 \otimes P(\lambda).$$

E'val E'vec Relations

Suppose there are F, G of full rank and $g, h \in \mathbb{C}^\ell$ s.t.

$$L(\lambda)F(\lambda) = g \otimes P(\lambda), \qquad E(\lambda)L(\lambda) = h^T \otimes P(\lambda).$$

- $F(\lambda)x \in \mathbb{C}^{\ell n}$ is a right e'vec of L with e'val λ iff $x \in \mathbb{C}^n$ is a right e'vec of P with e'val λ.
- If $w \in \mathbb{C}^{\ell n}$ is a left e'vec of L with e'val λ then $(g^* \otimes I_n)w \in \mathbb{C}^n$ is a left e'vec of P (if it's nonzero).
- If z is right e'vec of L with e'val λ then $(h^T \otimes I_n)z$ is a right e'vec of P with e'val λ (if it's nonzero).
- $E(\lambda)^* y$ is a left e'vec of L with e'val λ iff y is a left e'vec of P with e'val λ.

(Grammont, Higham, T. 2011)

Comparing κ_P to κ_L

$$\kappa_P(\lambda) = \frac{\left(\sum_{i=0}^{\ell} |\lambda|^i \, \|A_i\|_2\right) \|y\|_2 \|x\|_2}{|\lambda| \, |y^* P'(\lambda) x|}, \qquad \kappa_L(\lambda) = \frac{\left(|\lambda| \, \|X\|_2 + \|Y\|_2\right) \|w\|_2 \|z\|_2}{|\lambda| \, |w^* L'(\lambda) z|}.$$

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Lemma

Let (λ, x) be a finite e'pair of P and w be a left e'vec of L. Assume $L(\lambda)F(\lambda) = g \otimes P(\lambda)$, and $y = (g^* \otimes I)w \neq 0$. Then $z = F(\lambda)x$ is right e'vec of L, y is left e'vec of P, and

$$w^* L'(\lambda) z = y^* P'(\lambda) x.$$

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$$w^* L'(\lambda) z = y^* P'(\lambda) x.$$

Hence investigate size of

$$\frac{\kappa_L(\lambda)}{\kappa_P(\lambda)} = \frac{\left(|\lambda|\,\|X\|_2 + \|Y\|_2\right) \|w\|_2 \|z\|_2}{\left(\sum_{i=0}^{\ell} |\lambda|^i \|A_i\|_2\right) \|y\|_2 \|x\|_2} := \phi_L(\lambda) \text{ (growth factor)}$$

Growth Factor $\phi_L = \kappa_L(\lambda)/\kappa_P(\lambda)$

Obtained explicit approximations of ϕ_L for many L.

For example, when $P(\lambda) = Q(\lambda) = \lambda^2 M + \lambda D + K$, sufficient conditions for $\kappa_Q \approx \kappa_L$ are:

Linearization	Eigenvalue	Condition
C_1	No restriction	$\|M\|_2 \approx \|D\|_2 \approx \|K\|_2 \approx 1$
L_1	$\lvert\lambda\rvert \gtrsim 1$	$\rho \approx 1$
	$\lvert\lambda\rvert \ll 1$	"not available"
L_2	$\lvert\lambda\rvert \gtrsim 1$	"not available"
	$\lvert\lambda\rvert \ll 1$	$\rho \approx 1$

$$\rho = \max(\|M\|_2, \|D\|_2, \|K\|_2)/\min(\|M\|_2, \|K\|_2).$$

Bounding η_P/η_L

$$\eta_P(x,\lambda) = \frac{\|P(\lambda)x\|_2}{\left(\sum_{i=0}^{\ell} |\lambda^i| \|A_i\|_2\right)\|x\|_2}, \quad \eta_L(x,\lambda) = \frac{\|L(\lambda)z\|_2}{\left(|\lambda|\|X\|_2 + \|Y\|_2\right)\|z\|_2}.$$

Suppose

$$E(\lambda)L(\lambda) = h^T \otimes P(\lambda), \quad \in \mathbb{C}^{\ell} \quad \text{(left factorization)}$$

Let (λ, z) be approx e'pair of L and take $x = (h^T \otimes I_n)z$ as approx right e'vec of P.

Then $P(\lambda)x = E(\lambda)L(\lambda)z$ and

$$\frac{\eta_P(x,\lambda)}{\eta_L(z,\lambda)} \le \frac{|\lambda|\,\|X\|_2 + \|Y\|_2}{\sum_{i=0}^{\ell} |\lambda|^i \|A_i\|_2} \cdot \frac{\|E(\lambda)\|_2 \|z\|_2}{\|x\|_2}.$$

Sufficient conditions for $\eta_Q \approx \eta_L$

For $Q(\lambda) = \lambda^2 M + \lambda D + K$, let $m = \|M\|_2$, $d = \|D\|_2$, $k = \|K\|_2$,

$$\rho = \max(m, d, k)/\min(m, k), \qquad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \approx \begin{bmatrix} \lambda x \\ x \end{bmatrix}.$$

Linearization	E'val	E'vec	Condition
C_1	$\lvert\lambda\rvert \geq 1$	z_1	$d \leq m \approx k \approx 1$
	$\lvert\lambda\rvert \leq 1$	z_2	
L_1	$\lvert\lambda\rvert \geq 1$	z_1	$\rho \approx 1$
	$\lvert\lambda\rvert \leq 1$	z_2	$\rho \max\left(1, (m+d)\|K^{-1}\|_2\right) \approx 1$
L_2	$\lvert\lambda\rvert \geq 1$	z_1	$\rho \max\left(1, (d+k)\|M^{-1}\|_2\right) \approx 1$
	$\lvert\lambda\rvert \leq 1$	z_2	$\rho \approx 1$

Entirely consistent with conditions for $\kappa_Q(\lambda) \approx \kappa_L(\lambda)$.

Scaling $Q(\lambda) = \lambda^2 M + \lambda D + K$

Analysis shows importance of **scaling** QEPs/PEPs before computing e'vals via linearization.

- Eigenvalue parameter scaling:

$$\lambda = \gamma\mu, \quad \widetilde{P}(\mu) := \delta P(\gamma\mu).$$

- Does not affect sparsity of matrix coeffs.
- γ, δ chosen to improve growth factors $\kappa_L(\lambda)/\kappa_P(\lambda)$ (conditioning), $\eta_P(z_i, \lambda)/\eta_L(z, \lambda)$ (backward error).
- Has no effect on the eigenvalue conditioning for $P(\lambda)$ and backward error for $P(\lambda)$.
- Fan, Lin and Van Dooren scaling (2004) and tropical scaling (Gaubert & Sharify, 2010).

Fan, Lin and Van Dooren Scaling

If $\|K\|_2, \|M\|_2 \neq 0$, then $\gamma = \sqrt{\dfrac{\|K\|_2}{\|M\|_2}}$, $\delta = \dfrac{2}{\|K\|_2 + \|D\|_2 \gamma}$

minimize the maximum distance

$$\min_{\gamma,\delta} \max\{\| \underbrace{\delta K}_{\widetilde{K}} \|_2 - 1, \| \underbrace{\delta \gamma D}_{\widetilde{D}} \|_2 - 1, \| \underbrace{\delta \gamma^2 M}_{\widetilde{M}} \|_2 - 1\}.$$

Can show that if $\|D\|_2 \leq (\|M\|_2 \|K\|_2)^{1/2}$ then with this scaling and for companion linearizations $\widetilde{C}$,

$$\eta_Q \approx \eta_{\widetilde{C}}, \quad \kappa_Q \approx \kappa_{\widetilde{C}}.$$

Tropical Scalar Polynomials

- Let $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ be the **tropical semiring** with

$$a \oplus b = \max(a, b), \quad a \otimes b = a + b \quad \text{for all } a, b \in \mathbb{R} \cup \{-\infty\}.$$

- The piecewise affine function

$$p(x) = \bigoplus_{k=0}^{d} p_k \otimes x^{\otimes k} = \max_{0 \le k \le d} (p_k + kx), \quad p_k \in \mathbb{R} \cup \{-\infty\}$$

is a **tropical polynomial** of degree d.

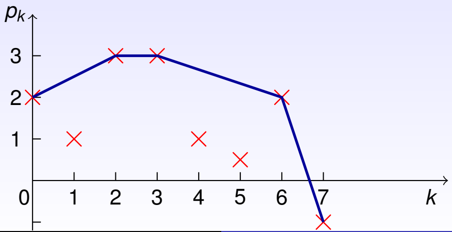
- The **tropical roots** of $p(x)$ are the points of nondifferentiability of $p(x)$.

Computation of Tropical Roots

Let $p(x) = \bigoplus_{k=0}^{d} p_k \otimes x^{\otimes k} = \max(p_0, p_1 + x, \dots, p_d + dx)$.

- Upper boundary of **convex hull** of (k, p_k), $k = 0\colon d$.
- **Tropical roots** are minus the **slopes** of the segments (Legendre-Fenchel duality).
- Horizontal **width** of segment gives **multiplicity**.

$p(x) = max(2, 1+x, 3+2x, 3+3x, 1+4x, \frac{1}{2}+5x, 2+6x, -1+7x)$ has roots $-\frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 3$.



Tropical Roots (cont.)

- Tropical roots can be computed in **linear time**.
- **Classical roots** of $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ can be **bounded in terms of tropical roots** of $p_{\mathrm{trop}}(x) = \bigoplus_{k=0}^{d} p_k \otimes x^{\otimes k}$ (Sharify, 11).
- Let r_1, r_2 be the **tropical roots** of

$$p_{\mathrm{trop}}(r) = \max(\log(\|K\|), \log(\|D\|) + r, \log(\|M\|) + 2r).$$

When $r_1 \gg r_2$ and M, D, K are well conditioned, $e^{r_1} \approx |\lambda_{\max}(Q)|$ and $e^{r_2} \approx |\lambda_{\min}(Q)|$, where $Q(\lambda) = \lambda^2 M + \lambda D + K$ (Gaubert & Sharify, 09).

Tropical Scaling

Let $p_{\text{trop}}(r) = \max(\log(\|K\|), \log(\|D\|) + r, \log(\|M\|) + 2r)$.

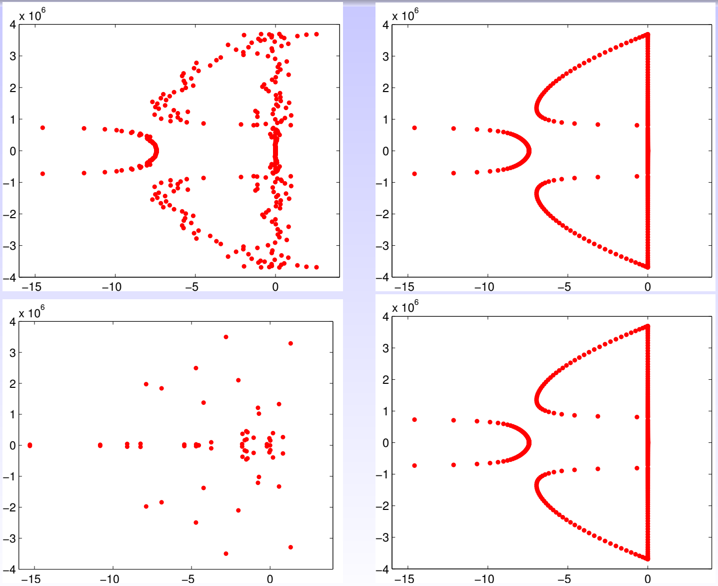
Convert $Q(\lambda)$ to $\delta Q(\mu\gamma)$, where

- $\gamma = e^r$, r is a tropical root of p_{trop},
- $\delta = e^{p_{\text{trop}}(r)}$.

- If $\|D\|_2^2 \le \|M\|_2\|K\|_2$ then $r_1 = r_2$ (one double root). Can show that $\eta_Q \approx \eta_{\widetilde{C}}$, $\kappa_Q \approx \kappa_{\widetilde{C}}$ for all e'vals.
- Otherwise, two distinct tropical roots $r_1 > r_2$. Can show that for companion linearizations $\widetilde{C}(\lambda)$, $\eta_Q \approx \eta_{\widetilde{C}}$,

  $\kappa_Q \approx \kappa_{\widetilde{C}}$ if $\begin{cases} r = r_1 \text{ and } |\lambda| \ge \gamma, \\ r = r_2 \text{ and } |\lambda| \le \gamma. \end{cases}$

Spectrum of C_1, L_2 before/after FLV Scaling



`quadeig` for $Q(\lambda) = \lambda^2 M + \lambda D + K$.

- Eigensolver for dense (small to medium size quadratics)—**`quadeig`**.
- Incorporates:
  - Appropriate choice of linearization: uses $\begin{bmatrix} D & -I \\ K & 0 \end{bmatrix} - \lambda \begin{bmatrix} -M & 0 \\ 0 & -I \end{bmatrix}$.
  - Deflation of 0 and ∞ eigenvalues.
  - Eigenvalue parameter scaling (FLV/tropical).
  - Careful recovery of the eigenvectors.
  - Backward stable when $\|D\| \lesssim (\|M\|\|K\|)^{1/2}$.
- MATLAB and Fortran implementations (NAG, LAPACK) [Hammarling, Munro, T. 2013].

Pseudospectra for Matrix Polynomials

Let $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ and $\Delta P(\lambda) = \sum_{j=0}^{\ell} \lambda^j \Delta A_j$.

We define the **ϵ-pseudospectrum** of P by

$$\Lambda_\epsilon(P) = \{\, \lambda \in \mathbb{C} : (P(\lambda) + \Delta P(\lambda))x = 0 \ \text{ for some } x \neq 0 \\ \text{and } \Delta P(\lambda) \text{ with } \|\Delta A_j\| \leq \epsilon\omega_j,\, j = 0\colon \ell \,\},$$

ω_j: nonneg. parameters, $\|\cdot\|$: subordinate matrix norm.

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ω_j: nonneg. parameters, $\|\cdot\|$: subordinate matrix norm.

Can obtain computable expressions

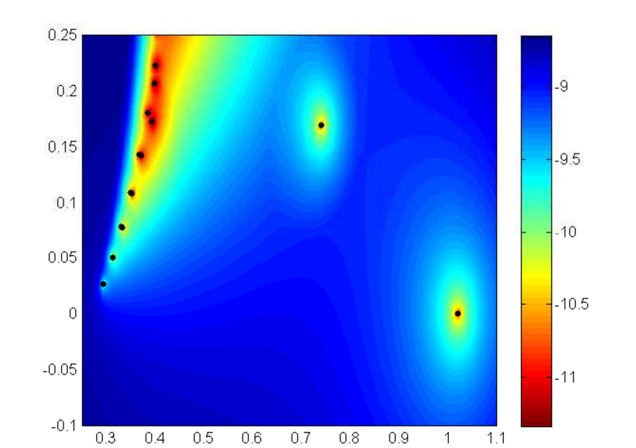
$$\begin{aligned}\Lambda_\epsilon(P) &= \{\, \lambda \in \mathbb{C} : \|P(\lambda)^{-1}\| \geq (\epsilon\, p(|\lambda|))^{-1} \,\} \\ &= \{\, \lambda \in \mathbb{C} : \eta_P(\lambda) \leq \epsilon \,\},\end{aligned}$$

where $p(|\lambda|) = \sum_{j=0}^{\ell} |\lambda|^j \omega_j$. (T. Higham, 01)

Pseudospectra of Orr-Sommerfeld Equation

Linearization of the incompressible Navier–Stokes eqns.

$$\left[\left(\frac{d^2}{dy^2}-\lambda^2\right)^2 - iR\left\{(\lambda U-\omega)\left(\frac{d^2}{dy^2}-\lambda^2\right)-\lambda U''\right\}\right]\phi = 0.$$



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