Matrix Polynomials in the Theory of Linear Control Systems

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Matrix Polynomials in Linear Control

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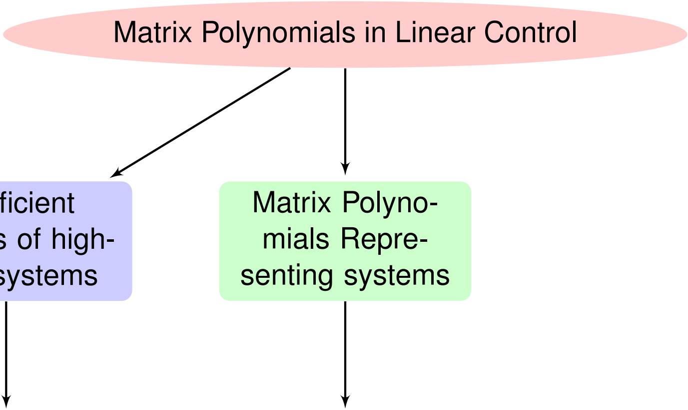
Coefficient Matrices of high-order systems

$$\begin{cases} T\left(\frac{\mathsf{d}}{\mathsf{d}t}\right)x(t) = U\left(\frac{\mathsf{d}}{\mathsf{d}t}\right)u(t) \\ y(t) = V\left(\frac{\mathsf{d}}{\mathsf{d}t}\right)x(t) + W\left(\frac{\mathsf{d}}{\mathsf{d}t}\right)u(t) \end{cases}$$



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$$\dot{x}(t) = Ax(t) + Bu(t)$$

Matrix Polynomials in Linear Control

- **Coefficient Matrices of high-order systems**

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- **Matrix Polynomials Representing systems**

$$\dot{x}(t) = Ax(t) + Bu(t)$$

- **Behaviours (Poldermann & Willems)**

$$P\left(\frac{\mathsf{d}}{\mathsf{d}t}\right)y(t) = Q\left(\frac{\mathsf{d}}{\mathsf{d}t}\right)u(t)$$

Polynomial System Matrices

$$\begin{cases} T\left(\dfrac{\mathsf{d}}{\mathsf{d}t}\right)x(t) = U\left(\dfrac{\mathsf{d}}{\mathsf{d}t}\right)u(t) \\ y(t) = V\left(\dfrac{\mathsf{d}}{\mathsf{d}t}\right)x(t) + W\left(\dfrac{\mathsf{d}}{\mathsf{d}t}\right)u(t) \end{cases}$$

Polynomial System Matrices

state vector

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Polynomial System Matrices

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input or control vector

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output or measurement vector

Polynomial System Matrices

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output or measurement vector

If $R(\lambda) = R_p\lambda^p + R_{p-1}\lambda^{p-1} + \cdots + R_1\lambda + R_0$ is a matrix polynomial:

$$R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) x(t) = R_p \frac{\mathrm{d}^p x(t)}{\mathrm{d}t^p} + R_{p-1}\frac{\mathrm{d}^{p-1} x(t)}{\mathrm{d}t^{p-1}} + \cdots + R_1 \frac{\mathrm{d}x(t)}{\mathrm{d}t} + R_0 x(t)$$

Polynomial System Matrices

state vector

input or control vector

$$
\begin{cases}
T\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) x(t) = U\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) u(t) \\
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$$

output or measurement vector

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$$

$$
\begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} \begin{bmatrix} \bar{x}(s) \\ -\bar{u}(s) \end{bmatrix} = \begin{bmatrix} 0 \\ -\bar{y}(s) \end{bmatrix}, \qquad \det T(s) \neq 0
$$

Polynomial System Matrices

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Polynomial System Matrix

Transfer Function Matrix

$\bar{y}(s) = V(s)\bar{x}(s) + W(s)\bar{u}(s)$ =

Transfer Function Matrix

$$T(s)\bar{x}(s) = U(s)\bar{u}(s)$$

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Transfer Function Matrix

$$T(s)\bar{x}(s) = U(s)\bar{u}(s)$$

$$\bar{y}(s) = V(s)\bar{x}(s) + W(s)\bar{u}(s) = \left(V(s)T(s)^{-1}U(s) + W(s)\right)\bar{u}(s)$$

Transfer Function Matrix

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When do two polynomial system matrices
yield the same Transfer Function Matrix?

Strict System Equivalence

Unimodular:
$\det = c \neq 0$

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I_p \end{bmatrix} \begin{matrix} \\ n \\ p \end{matrix} \begin{bmatrix} \overset{n}{T_1(s)} & \overset{m}{U_1(s)} \\ -V_1(s) & W_1(s) \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} T_2(s) & U_2(s) \\ -V_2(s) & W_2(s) \end{bmatrix}$$

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$$\Downarrow$$

$$V_1(s)T_1(s)^{-1}U_1(s) + W_1(s) = V_2(s)T_2(s)^{-1}U_2(s) + W_2(s)$$

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$$\Downarrow \qquad \Uparrow \quad ?$$

$$V_1(s)T_1(s)^{-1}U_1(s) + W_1(s) = V_2(s)T_2(s)^{-1}U_2(s) + W_2(s)$$

Coprime Matrix Polynomials

$$\begin{cases} A(s) = \widetilde{A}(s)\, C(s) \\ \\ B(s) = \widetilde{B}(s)\, C(s) \end{cases}$$

Right Common Factor

$A(s),\ B(s)$ **right coprime** $\Leftrightarrow$ $C(s)$ unimodular

$A(s)$, $B(s)$ right coprime $\Leftrightarrow \begin{bmatrix} A(s) \\ B(s) \end{bmatrix} \overset{e}{\sim} \begin{bmatrix} I_n \\ 0 \end{bmatrix}$

$A(s)$, $B(s)$ left coprime $\Leftrightarrow \begin{bmatrix} A(s) & B(s) \end{bmatrix} \overset{e}{\sim} \begin{bmatrix} I_n & 0 \end{bmatrix}$

Coprimeness and Strict System Equivalence

If $G(s) \in \mathbb{F}(s)^{p\times m}$, $(T(s), U(s), V(s), W(s))$ **Realization** of $G(s)$ if $G(s) = W(s) + V(s)T(s)^{-1}U(s)$. **order**= $\deg(\det T(s))$

$(T(s), U(s), V(s), W(s))$ realization of **least** order $\Leftrightarrow$
$(T(s), U(s))$ left coprime and $(T(s), V(s))$ right coprime

If $P_1(s) = \begin{bmatrix} T_1(s) & U_1(s) \\ -V_1(s) & W_1(s) \end{bmatrix}$, $P_2(s) = \begin{bmatrix} T_2(s) & U_2(s) \\ -V_2(s) & W_2(s) \end{bmatrix}$ polynomial system matrices of *least order*

$$P_1(s) \overset{s.s.e.}{\sim} P_2(s)$$
$$\Updownarrow$$
$$V_1(s)T_1(s)^{-1}U_1(s) + W_1(s) = V_2(s)T_2(s)^{-1}U_2(s) + W_2(s)$$

Systems in State-Space Form

$$(\Sigma) \quad \left\{ \begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{array} \right. \quad \rightarrow \quad P(s) = \begin{bmatrix} sI_n - A & B \\ -C & 0 \end{bmatrix}$$

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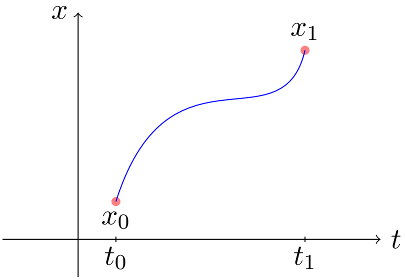
Controllability in $[t_0, t_1]$: $\forall x_0, x_1 \in \mathbb{R}^n$, $\exists u$ defined in $[t_0, t_1]$ such that the solution of the I.V.P. $\dot{x}(t) = Ax(t) + Bu(t)$, $x(t_0) = x_0$, satisfies $x(t_1) = x_1$.



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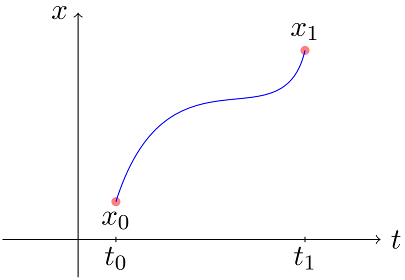

(Σ) controllable $\leftrightarrow$ (A, B) controllable

Controllability, Observability and Coprimeness

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(A, B) controllable is equivalent to:

- $\operatorname{rank} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n$, or
- $sI_n - A$ and B are left coprime $\left(\begin{bmatrix} sI_n - A & B \end{bmatrix} \overset{e}{\sim} \begin{bmatrix} I_n & 0 \end{bmatrix} \right)$

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Observability in $[t_0, t_1]$: The value of y in $[t_0, t_1]$ determines the state at t_0, $x(t_0)$, and so the vector function $x(t)$ in $[t_0, t_1]$.

(Σ) observable $\leftrightarrow$ (A, C) **observable** $\equiv$ (A^T, C^T) controllable.

Controllability, Observability and Coprimeness

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(Σ) observable $\leftrightarrow$ (A, C) **observable** $\equiv$ (A^T, C^T) controllable.

$P(s) = \begin{bmatrix} sI_n - A & B \\ -C & 0 \end{bmatrix}$ is of least order if and only if (A, B) controllable and (A, C) observable.

Transfer Function Matrix

From now on:

$$(\Sigma) \qquad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = I_n x(t) \end{cases} \qquad \rightarrow \quad P(s) = \begin{bmatrix} sI_n - A & B \\ -I_n & 0 \end{bmatrix}$$

Transfer Function Matrix: $G(s) = (sI_n - A)^{-1}B \underset{s \to \infty}{\longrightarrow} 0$

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$\mathrm{lcm}\{$denominators of $G(s)\}$

$$G(s) = \widetilde{N}(s)(\, d(s) \, I_n)^{-1}$$

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Remove right common factors from $\widetilde{N}(s)$ and $d(s)I_n$

$$G(s) = \widetilde{N}(s)(\,d(s)\,I_n)^{-1} = N(s)D(s)^{-1}$$

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If (A, B) controllable

$$\begin{bmatrix} sI_n - A & B \\ -I_n & 0 \end{bmatrix} \overset{sse}{\sim} \left[\begin{array}{cc|c} I_{n-m} & 0 & 0 \\ 0 & D(s) & I_m \\ \hline 0 & -N(s) & 0 \end{array}\right] \qquad (n \geq m)$$

Polynomial Matrix Representations

$$\begin{bmatrix} U(s) & 0 \\ X(s) & I_n \end{bmatrix} \begin{bmatrix} sI_n - A & B \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} V(s) & Y(s) \\ 0 & I_m \end{bmatrix} = \left[\begin{array}{cc|c} I_{n-m} & 0 & 0 \\ 0 & D(s) & I_m \\ \hline 0 & -N(s) & 0 \end{array} \right]$$

$$\Downarrow$$

$$U(s) \begin{bmatrix} sI_n - A & B \end{bmatrix} \begin{bmatrix} V(s) & Y(s) \\ 0 & I_m \end{bmatrix} = \left[\begin{array}{cc|c} I_{n-m} & 0 & 0 \\ 0 & D(s) & I_m \end{array} \right] \qquad (\star)$$

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$$\Uparrow \quad \Downarrow$$

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$X(s) = \begin{bmatrix} V_1(s) & Y(s) \end{bmatrix} U(s)$, $N(s) = -Y(s)D(s) + V_2(s)$, where $V(s) = \begin{bmatrix} V_1(s) & V_2(s) \end{bmatrix}$, V_2 $n \times m$

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A **Polynomial Matrix Representation** of a controllable system (A, B) is any non-singular Matrix Polynomial $D(s)$ such that $(\star)$ is satisfied for some unimodular matrices $U(s)$, $V(s)$ and some matrix $Y(s)$.

Some Consequences of the Definition. I

- If $D(s)$ is a PMR of (A, B), $D(s)W(s)$ is also a PMR of (A, B) for any unimodular $W(s)$.

$$U(s)\begin{bmatrix} sI_n - A & B \end{bmatrix}\begin{bmatrix} V(s) & Y(s) \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} I_{n-m} & 0 & 0 \\ 0 & D(s) & I_m \end{bmatrix}$$

$$\Downarrow$$

$$U(s)\begin{bmatrix} sI_n - A & B \end{bmatrix}\begin{bmatrix} V(s)\widetilde{W}(s) & Y(s) \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} I_{n-m} & 0 & 0 \\ 0 & D(s)W(s) & I_m \end{bmatrix}.$$

$\widetilde{W}(s) = \mathrm{Diag}(I_{n-m}, W(s))$.

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$$\Downarrow$$
$$U(s)\begin{bmatrix} sI_n - A & B \end{bmatrix}\begin{bmatrix} V(s)\widetilde{W}(s) & Y(s) \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} I_{n-m} & 0 & 0 \\ 0 & D(s)W(s) & I_m \end{bmatrix}.$$

$\widetilde{W}(s) = \mathrm{Diag}(I_{n-m}, W(s))$.

If $D_1(s)$, $D_2(s)$ are PMRs of (A_1, B_1) and (A_2, B_2) :

$$D_2(s) = D_1(s)W(s) \quad \Leftrightarrow \quad (A_2, B_2) = (P^{-1}A_1P, P^{-1}B_1).$$

$W(s)$ unimodular, P invertible.

Some Consequences of the Definition. II

- If $D(s) = D_\ell s^\ell + D_{\ell-1} s^{\ell-1} + \cdots + D_1 s + D_0$

$$(sI_n - A)^{-1}B = N(s)D(s)^{-1} \;\Leftrightarrow\; (sI_n - A)^{-1}BD(s) = N(s)$$
$$\Leftrightarrow\; A^\ell B D_\ell + A^{\ell-1} B D_{\ell-1} + \cdots + ABD_1 + BD_0 = 0 \quad (\star\star)$$

(A, B) controllable and $(\star\star) \Leftrightarrow D(s)$ PMR of (A, B)

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- $sI_n - A$ is a linearization of $D(s)$

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Given $D(s)$, non-singular, is there, for any linearization $sI_n - A$ of $D(s)$, a control matrix B such that (A, B) is controllable and $D(s)$ is a PMR of (A, B)?

Controllability indices

Assume (A, B) controllable: $\operatorname{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n$

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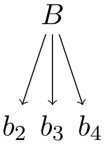

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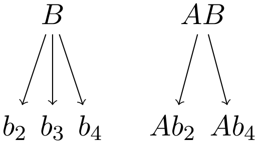

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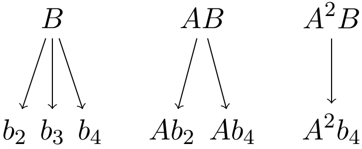

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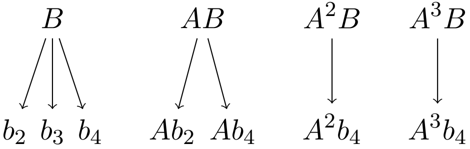

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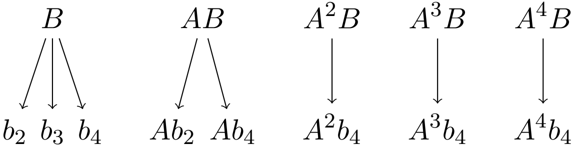

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$B \quad AB \quad A^2B \quad A^3B \quad A^4B \quad A^5B \quad A^6B \quad A^7B$

$B \to b_2\ b_3\ b_4$

$AB \to Ab_2\ Ab_4$

$A^2B \to A^2b_4$

$A^3B \to A^3b_4$

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$B \to b_2,\ b_3,\ b_4 \qquad AB \to Ab_2,\ Ab_4 \qquad A^2B \to A^2b_4 \qquad A^3B \to A^3b_4 \qquad A^4B \to A^4b_4$

$\ell_1 = 0,\ \ell_2 = 2,\ \ell_3 = 1,\ \ell_4 = 5,\ \ell_5 = 0$

$k_1 = 5,\ k_2 = 2,\ k_3 = 1,\ k_4 = 0,\ k_5 = 0$

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$$B \quad AB \quad A^2B \quad A^3B \quad A^4B \quad A^5B \quad A^6B \quad A^7B$$

$$\downarrow$$

$$b_2\ b_3\ b_4 \quad Ab_2\ Ab_4 \quad A^2b_4 \quad A^3b_4 \quad A^4b_4$$

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Controllability Indices

Controllability indices of (A, B)= minimal indices of $s\begin{bmatrix} I_n & 0 \end{bmatrix} - \begin{bmatrix} A & B \end{bmatrix}$

Matlab

PMRs and Controllability indices

unordered controllability indeces

$$D(s) = D_{hc} \begin{bmatrix} s^{\ell_1} & 0 & \cdots & 0 \\ 0 & s^{\ell_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s^{\ell_m} \end{bmatrix} + D_{lc}(s)$$

$\det D_{hc} \neq 0$

degree of i-th column $\leq \ell_i$

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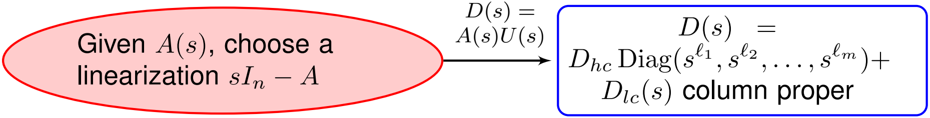
- Matrix polynomials with this property are called **column proper** or **column reduced**

Let $A(s) \in \mathbb{F}[s]^{m\times m}$ be a non-singular matrix polynomial. Then there is $U(s) \in \mathbb{F}[s]^{m\times m}$, unimodular, such that $D(s) = A(s)U(s)$ is a column proper matrix. In general, $D(s)$ is not unique but all have the same column degrees up to reordering.

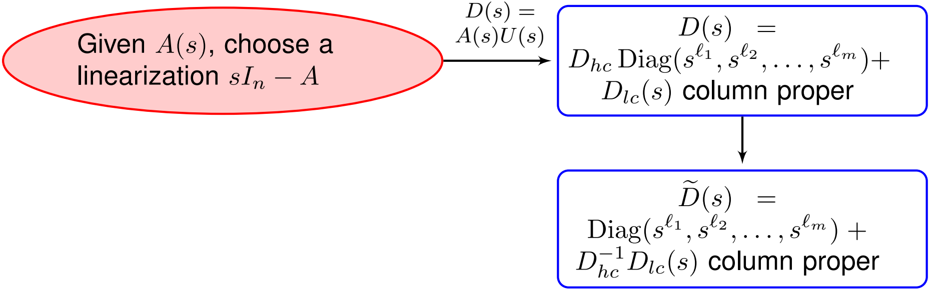
Column Proper Matrix Polynomials

Given $A(s)$, choose a linearization $sI_n - A$

Column Proper Matrix Polynomials



Column Proper Matrix Polynomials



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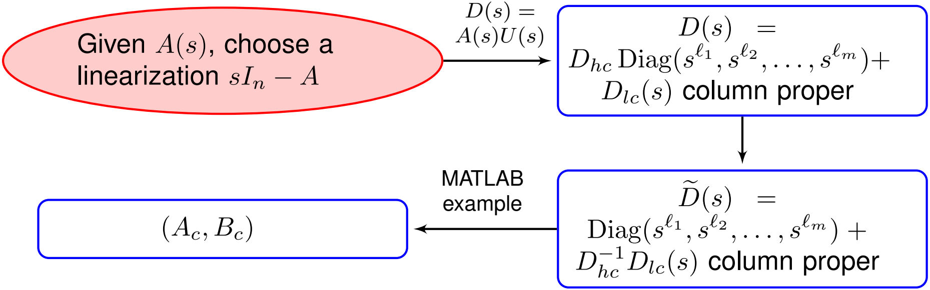
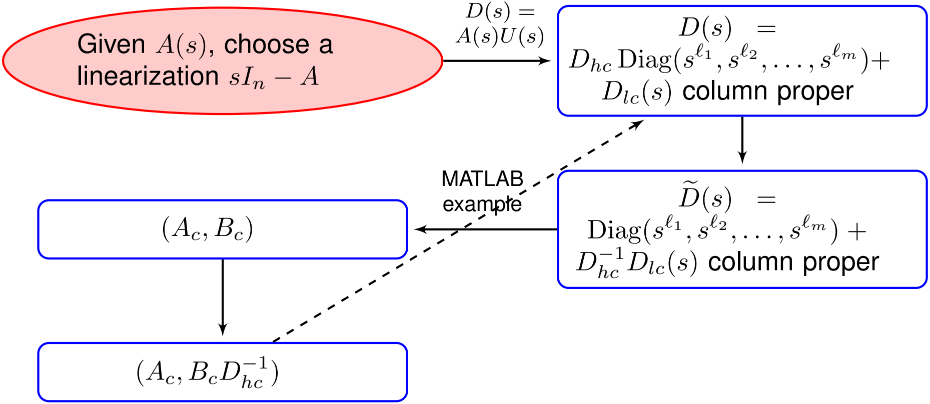


Column Proper Matrix Polynomials



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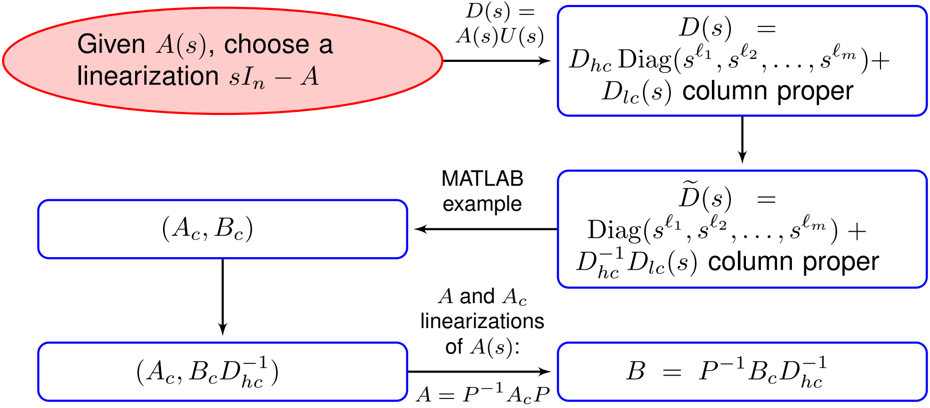

$A(s)$ is a Polynomial Matrix Representation of (A, B) and $\ell_1, \ldots \ell_m$ are its (unordered) controllability indices.

Wiener-Hopf Factorization and Indices

$A_1(s), A_2(s) \in \mathbb{F}[s]^{m\times n}$ Wiener-Hopf equivalent (*at* ∞ *on the left*):

$$A_2(s) = B(s)\, A_1(s)\, U(s)$$

Biproper: $\lim_{s\to\infty} B(s)$ invertible

Unimodular: $\lim_{s\to a} U(s)$ invertible, $\forall a \in \mathbb{C}$

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$$\begin{aligned} A(s)U(s) &= D_{hc}\,\mathrm{Diag}(s^{\ell_1},\ldots,s^{\ell_m}) + D_{lc}(s) \\ &= \underbrace{[D_{hc} + D_{lc}(s)\,\mathrm{Diag}(s^{-\ell_1},\ldots,s^{-\ell_m})]}_{B(s)\in\mathbb{F}_{pr}(s)^{m\times m}}\,\mathrm{Diag}(s^{\ell_1},\ldots,s^{\ell_m}) \end{aligned}$$

$$\lim_{s\to\infty} B(s) = D_{hc} \quad \text{invertible}$$

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$$A(s) \overset{W-H}{\sim} \mathrm{Diag}(s^{k_1}, s^{k_2}, \dots, s^{k_m})$$

$k_1 \geq k_2 \geq \cdots \geq k_m$= Wiener-Hopf factorization indices of $A(s)$

Brunovsky-Kronecker canonical form

$\mathrm{Diag}(s^{k_1}, s^{k_2}, \ldots, s^{k_m})$ is a PMR of (A_c, B_c)

$$A_c = \mathrm{Diag}\left\{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{F}^{k_i \times k_i}\right\}, \quad B_c = \mathrm{Diag}\left\{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{F}^{k_i \times 1}\right\}, \; 1 \le i \le m$$

$$s\begin{bmatrix} I_n & 0 \end{bmatrix} - \begin{bmatrix} A_c & B_c \end{bmatrix} \overset{se}{\sim} \mathrm{Diag}\left\{\begin{bmatrix} s & 1 & 0 & \cdots & 0 & 0 \\ 0 & s & 1 & \cdots & 0 & 0 \\ 0 & 0 & s & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s & 1 \end{bmatrix} \in \mathbb{F}^{k_i \times (k_i+1)} : 1 \le i \le m\right\}$$

Kronecker canonical form

Brunovsky-Kronecker canonical form

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Kronecker canonical form

(A_c, B_c) = system in Brunovsky form

Feedback equivalence

$$(A, B) \text{ has controllability indices } k_1 \geq k_2 \geq \cdots \geq k_m$$

$$\Updownarrow$$

$$s\begin{bmatrix} I_n & 0 \end{bmatrix} - \begin{bmatrix} A & B \end{bmatrix} \overset{se}{\sim} s\begin{bmatrix} I_n & 0 \end{bmatrix} - \begin{bmatrix} A_c & B_c \end{bmatrix}$$

$$\Updownarrow$$

$$P(s\begin{bmatrix} I_n & 0 \end{bmatrix} - \begin{bmatrix} A & B \end{bmatrix})T = s\begin{bmatrix} I_n & 0 \end{bmatrix} - \begin{bmatrix} A_c & B_c \end{bmatrix}$$

$$\Updownarrow$$

$$P\begin{bmatrix} A & B \end{bmatrix}\begin{bmatrix} P^{-1} & 0 \\ R & Q \end{bmatrix} = \begin{bmatrix} A_c & B_c \end{bmatrix}$$

$$\Updownarrow$$

$$(A_c, B_c) = (PAP^{-1} + PBF, PBQ)$$

Feedback equivalence

(A, B) has controllability indices $k_1 \geq k_2 \geq \cdots \geq k_m$

$$\Updownarrow$$

$$s\begin{bmatrix} I_n & 0 \end{bmatrix} - \begin{bmatrix} A & B \end{bmatrix} \overset{se}{\sim} s\begin{bmatrix} I_n & 0 \end{bmatrix} - \begin{bmatrix} A_c & B_c \end{bmatrix}$$

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$$\Updownarrow$$

$$(A_c, B_c) = (PAP^{-1} + PBF, PBQ)$$

Feedback equivalence

Any controllable system is feedback equivalent to a system in Brunovsky form

Summarizing

$D(s)$ is a Polynomial Matrix Representation of a controllable (A, B)

$$\Updownarrow$$

$$U(s)\begin{bmatrix} sI_n - A & B \end{bmatrix}\begin{bmatrix} V(s) & Y(s) \\ 0 & I_m \end{bmatrix} = \left[\begin{array}{cc|c} I_{n-m} & 0 & 0 \\ 0 & D(s) & I_m \end{array}\right]$$

$$\Updownarrow$$

$(sI_n - A)^{-1}B = N(s)D(s)^{-1}, \quad N(s), D(s)$ right coprime

$$\Updownarrow$$

$$A^\ell B D_\ell + A^{\ell-1} B D_{\ell-1} + \cdots + ABD_1 + BD_0 = 0$$

$$(D(s) = D_\ell s^\ell + D_{\ell-1} s^{\ell-1} + \cdots + D_0)$$

Summarizing

$D(s)$ is a Polynomial Matrix Representation of a controllable (A, B)

$$\Updownarrow$$

$$U(s)\begin{bmatrix} sI_n - A & B \end{bmatrix}\begin{bmatrix} V(s) & Y(s) \\ 0 & I_m \end{bmatrix} = \left[\begin{array}{cc|c} I_{n-m} & 0 & 0 \\ 0 & D(s) & I_m \end{array}\right]$$

$$\Updownarrow$$

$$(sI_n - A)^{-1}B = N(s)D(s)^{-1}, \quad N(s), D(s) \text{ right coprime}$$

$$\Updownarrow$$

$$A^\ell BD_\ell + A^{\ell-1}BD_{\ell-1} + \cdots + ABD_1 + BD_0 = 0$$

$$(D(s) = D_\ell s^\ell + D_{\ell-1}s^{\ell-1} + \cdots + D_0)$$

(A, B)	$D(s)$
$(P^{-1}AP, P^{-1}B)$ Similarity	$D(s)U(s)$ Right Equivalence
$(P^{-1}AP + P^{-1}BF, P^{-1}BQ)$ Feedback Equivalence	$B(s)D(s)U(s)$ Wiener-Hopf Equivalence
Controllability Indices	Left Wiener-Hopf indices

Matrix Polynomials with non-singular Leading Coefficient

non-singular

$$D(s) = D_\ell s^\ell + D_{\ell-1}s^{\ell-1} + \cdots + D_1 s + D_0$$

$B(s) = D_\ell + D_{\ell-1}s^{-1} + \cdots + D_1 s^{-\ell+1} + D_0 s^{-\ell}$ biproper ($\lim_{s\to\infty} B(s) = D_\ell$)

$B(s)^{-1}D(s) = s^\ell I_m \Rightarrow (\overbrace{\ell, \ell, \ldots, \ell}^{m}) =$ Wiener-Hopf indices of $D(s)$

Matrix Polynomials with non-singular Leading Coefficient

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$B(s)^{-1}D(s) = s^\ell I_m \Rightarrow (\overbrace{\ell, \ell, \ldots, \ell}^{m}) =$ Wiener-Hopf indices of $D(s)$

$D(s)$ Polynomial Matrix Representation of (A, B)

$$\Updownarrow$$

$\operatorname{rank}\begin{bmatrix} B & AB & \cdots & A^{\ell m-1}B \end{bmatrix} = \ell m$ and

$$A^\ell B D_\ell + A^{\ell-1}BD_{\ell-1} + \cdots + ABD_1 + BD_0 = 0$$

Since $(\overbrace{\ell, \ell, \ldots, \ell}^{m}) =$ Controllability indices of (A, B)

$$\operatorname{rank}\begin{bmatrix} B & AB & \cdots & A^{\ell m-1}B \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B & AB & \cdots & A^{\ell-1}B \end{bmatrix}$$

(A, B)= Standard Pair of $D(s)$

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