Nonnegative combined matrices. Relation with sign regular matrices

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(Joint work with Rafael Bru, M. Teresa Gassó and Máximo Santana)

Jornadas sobre matrices totalmente positivas y totalmente negativas,
ALAMA-CIEM, Castro Urdiales,
5–6 Marzo, 2015

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Combined matrices, Applications

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- *T.J. McAvoy, Some results on dynamic interaction analysis of complex control systems, Ind. Eng. Chem. Process Des. Dev. 22 (1) (1983) 42—49*

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Combined matrices, Doubly stochastic matrices

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$$C(A) = (c_{ij}) \rightarrow \sum_j c_{ij} = 1 \quad \forall i \quad \text{and} \quad \sum_i c_{ij} = 1 \quad \forall j$$

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R. Bru, M.T. Gassó, I. Giménez and M. Santana, Nonnegative combined matrices, JAM-2014, ID 182354

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Combined matrices, Mathematical aspects

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Despite the fact that "Combined matrix" denomination was firstly proposed in

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its study (and the one referred to Hadarmard product of different classes of matrices) had been started many years before, being continued to the present

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Fiedler & Pták 1967, Charles R. Johnson 1974 & 1977, Fiedler & Markham 1986, Johnson & Shapiro 1986

M. Fiedler and T.L. Markham, Combined matrices in special classes of matrices, LAA-435 (2011) 1945–1955

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Sign Regular Matrices

We consider square, real and nonsingular matrices

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Sign Regular Matrices

We consider square, real and nonsingular matrices

Signature

A k-vector of signs $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ with $\varepsilon_j \in \{+1, -1\}$ for $j = 1, \ldots, k$, is called a signature.

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Sign Regular Matrix

An $n \times n$ matrix A is **sign regular of order** k, $k \leq n$, with signature $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ if for each $j = 1, \ldots, k$, all its minors of order j have the same nonstrict sign that coincides with ε_j.
If $k = n$, matrix A is called **sign regular**.

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- A **TP** matrix is a sign regular matrix with signature $\varepsilon = (1, 1, \ldots, 1)$.

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- A **TP** matrix is a sign regular matrix with signature $\varepsilon = (1, 1, \dots, 1)$.
- A **t.n.p.** matrix is a sign regular matrix with signature $\varepsilon = (-1, -1, \dots, -1)$.

Sign regular matrices studies

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"The interest of nonsingular sign regular matrices comes from their characterizations as variation-diminishing linear maps: the number of sign changes in the consecutive components of the image of a vector is bounded above by the number of sign changes in the consecutive components of the signature."

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Koev & Dopico-2007, Cantó, Ricarte & Urbano-2012, Cortés & Peña-2008, R. Huang-2013, Alonso, Peña & Serrano-2015

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Questions

- If A is a sign regular matrix, could be $C(A) \geq 0$?

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- If A is a sign regular matrix, could be $C(A) \geq 0$?
- How is the zero pattern of $C(A)$?

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Questions

- If A is a sign regular matrix, could be $C(A) \geq 0$?
- How is the zero pattern of $C(A)$?
- What coordinates of the A signature determine the solution of the above questions ?

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Previous Results

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Theorem (Signature of $SA^{-1}S$)

Let A be a nonsingular sign regular matrix with signature $\varepsilon = \varepsilon(A)$ and $S = \mathrm{diag}(1, -1, 1, -1, \ldots, (-1)^{n-1})$. Then the matrix $SA^{-1}S$ is also sign regular with signature $\varepsilon' = \varepsilon(SA^{-1}S)$ such that $\varepsilon'_i = \varepsilon_i \varepsilon_{n-i}$ (with the convention $\varepsilon_j = 1$ when $j = 0$)

(A is SR $\leftrightarrow$ $SA^{-1}S$ is SR)

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$$S|B|S = \begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ & \cdots & & & \end{bmatrix} \quad \rightarrow \quad SA^{-1}S = \left[\frac{A(i|j)}{\det A}\right]$$

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Note that $\varepsilon(SA^{-T}S) = \varepsilon(SA^{-1}S)$

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Previous Results

Lemma (Zero pattern of A related to ε_2)

Let A be an $n \times n$ nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$

1. *If $\varepsilon_2 = 1$, then*
    1. $a_{11} \neq 0, a_{22} \neq 0, \ldots, a_{nn} \neq 0,$
    2. $a_{ij} = 0, \quad i > j \quad \Rightarrow a_{tl} = 0,$ *for all* $t \geq i$ *and* $l \leq j,$
    3. $a_{ij} = 0, \quad i < j \quad \Rightarrow a_{tl} = 0,$ *for all* $t \leq i$ *and* $l \geq j.$
2. *If $\varepsilon_2 = -1$, then*
    1. $a_{1n} \neq 0, a_{2,n-1} \neq 0, \ldots, a_{n1} \neq 0,$
    2. $a_{ij} = 0, \quad j > n - i + 1 \quad \Rightarrow a_{tl} = 0,$ *for all* $t \geq i$ *and* $l \geq j,$
    3. $a_{ij} = 0, \quad j < n - i + 1 \quad \Rightarrow a_{tl} = 0,$ *for all* $t \leq i$ *and* $l \leq j.$

R. Huang, Nonsingular almost strictly sign regular matrices, LAA-436 (2012)

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$\varepsilon_2 = 1$:

$$A = \begin{bmatrix} \bullet & & & & & & \\ & \bullet & & & & & \\ & & \bullet & & 0 & x_1 & \\ & & & \bullet & & & \\ & & & & \bullet & y_1 & \\ & 0 & & & & \bullet & \\ & x_2 & & & & y_2 & \bullet \end{bmatrix}$$

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$\varepsilon_2 = 1$:

$$A = \begin{bmatrix} \bullet & & & & 0 & \cdots & 0 \\ & \bullet & & & \vdots & \cdots & \vdots \\ & & \bullet & & 0 & \cdots & 0 \\ & & & \bullet & & & \\ & & & & \bullet & & \\ 0 & 0 & & & & \bullet & \\ 0 & 0 & & & & & \bullet \end{bmatrix}$$

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$\varepsilon_2 = -1$:

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Lemmas

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Lemmas

Lemma (1 : Zero pattern of $C(A) \geq 0$)

If A is an $n \times n$ nonsingular sign regular matrix with signature ε and $C(A) \geq 0$.

- *If $\varepsilon_1 \varepsilon_{n-1} \varepsilon_n = 1$, then $C(A)$ has the following zero pattern*

$$\begin{bmatrix} * & 0 & * & \cdots & * & 0 \\ 0 & * & 0 & \cdots & 0 & * \\ * & 0 & * & \cdots & * & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & 0 & * & \cdots & * & 0 \\ 0 & * & 0 & \cdots & 0 & * \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} * & 0 & * & \cdots & 0 & * \\ 0 & * & 0 & \cdots & * & 0 \\ * & 0 & * & \cdots & 0 & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & 0 & \cdots & * & 0 \\ * & 0 & * & \cdots & 0 & * \end{bmatrix},$$

if $n = 2k$ and if $n = 2k - 1$ $(k \in \mathbb{N})$, respectively.

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Lemmas

Lemma ($2: C(A) = \pm A \circ (SA^{-T}S)$)

Let A be an $n \times n$ nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$.

1. *If $\varepsilon_1 \varepsilon_{n-1} \varepsilon_n = 1$ and $C(A) \geq 0$, then $C(A) = A \circ (SA^{-T}S)$.*
2. *If $\varepsilon_1 \varepsilon_{n-1} \varepsilon_n = -1$ and $C(A) \geq 0$, then $C(A) = -A \circ (SA^{-T}S)$.*

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Lemmas

Lemma $(2: C(A) = \pm A \circ (SA^{-T}S))$

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$C(A) = [c_{ij}], \quad A \circ (SA^{-1}S) = [m_{ij}] \quad \rightarrow \quad c_{ij} = \pm m_{ij}$

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Lemmas

Lemma ($2: C(A) = \pm A \circ (SA^{-T}S)$)

Let A be an $n \times n$ nonsingular sign regular matrix with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$.

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$C(A) = [c_{ij}], \quad A \circ (SA^{-1}S) = [m_{ij}] \quad \rightarrow \quad c_{ij} = \pm m_{ij}$
and negative ones are null (by $C(A) \geq 0$ and Lemma 1)

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Lemmas

Lemma (3 : Zero pattern of $C(A)$ if $\varepsilon_2\varepsilon_{n-2}\varepsilon_n = -1$)

Let A be an $n \times n$ nonsingular sign regular matrix with signature ε such that $\varepsilon_2\varepsilon_{n-2}\varepsilon_n = -1$.
$C(A) \geq 0 \Rightarrow c_{ij} = 0$ *for all i, j such that $j \neq i$ and $j \neq n - i + 1$.*

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$$C(A) = \begin{bmatrix} \bullet & 0 & \cdots & & \cdots & 0 & \bullet \\ 0 & \bullet & 0 & \cdots & 0 & \bullet & 0 \\ 0 & 0 & \bullet & & \bullet & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \bullet & \cdots & \bullet & 0 & 0 \\ 0 & \bullet & 0 & \cdots & 0 & \bullet & 0 \\ \bullet & 0 & \cdots & & \cdots & 0 & \bullet \end{bmatrix}$$

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$C(A) \geq 0 \Rightarrow c_{ij} = 0$ for all i, j such that $j \neq i$ and $j \neq n-i+1$.

$c_{ij} \neq 0 \quad \rightarrow \quad c_{i,j-1} = 0$ by Lemma 1

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$c_{ij} \neq 0 \quad \rightarrow \quad c_{i,j-1} = 0$ by Lemma 1
$a_{i,j-1} = 0 \quad \rightarrow \quad a_{ij} = 0$ by Huang Lemma (A is sign regular) $\quad \rightarrow$ Contradiction !)

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$c_{ij} \neq 0 \quad \rightarrow \quad c_{i,j-1} = 0$ by Lemma 1
$a_{i,j-1} = 0 \quad \rightarrow \quad a_{ij} = 0$ by Huang Lemma (A is sign regular) $\rightarrow$ Contradiction !)
$A_{i,j-1} = 0 \quad \rightarrow \quad A_{ij} = 0$ by Huang Lemma ($SA^{-1}S$ is sign regular) $\rightarrow$ Contradiction !)

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Lemmas

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Lemmas

Lemma (4 : $C(A) = I$ or J when n is odd and ...)

Let A be an $n \times n$ nonsingular sign regular matrix with signature ε and $C(A) \geq 0$. If $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = 1$, $\varepsilon_2\varepsilon_{n-2}\varepsilon_n = -1$ and $n = 2k-1$ for some $k \in \mathbb{N}$, then $C(A)$ is either diagonal or antidiagonal.

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Lemma (4 : $C(A) = I$ or J when n is odd and ...)

Let A be an $n \times n$ nonsingular sign regular matrix with signature ε and $C(A) \geq 0$. If $\varepsilon_1 \varepsilon_{n-1} \varepsilon_n = 1$, $\varepsilon_2 \varepsilon_{n-2} \varepsilon_n = -1$ and $n = 2k - 1$ for some $k \in \mathbb{N}$, then $C(A)$ is either diagonal or antidiagonal.

Supposing $c_{1n} \neq 0$, then $a_{1n} \neq 0$ $\rightarrow$ $a_{i,j} \neq 0$ for $1 \leq j$ by Huang Lemma.

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Lemmas

Lemma (4 : $C(A) = I$ or J when n is odd and ...)

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$c_{i,n-i} = 0$ by Lemma 1, then $A_{i,n-i} = 0$ for $i < \dfrac{n+1}{2}$ $\rightarrow$ $A_{ii} = 0$ for $i < \dfrac{n+1}{2}$

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Theorems

A: $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = 1$	A.1: $\varepsilon_2 = 1, \varepsilon_{n-2}\varepsilon_n = 1$	$C(A) \geq 0 \Leftrightarrow C(A) = I$
	A.2: $\varepsilon_2 = -1, \varepsilon_{n-2}\varepsilon_n = -1$	$n = 2k,\ C(A)$ is not nonnegative $n = 2k-1,\ C(A) \geq 0 \Leftrightarrow C(A) = J$
	A.3: $\varepsilon_2(\varepsilon_{n-2}\varepsilon_n) = -1$	$n = 2k,\ C(A) \geq 0 \Leftrightarrow C(A)$=I $n = 2k-1$ $a_{1n} \neq 0,\ C(A) \geq 0 \Leftrightarrow C(A) = J$ $a_{11} \neq 0,\ C(A) \geq 0 \Leftrightarrow C(A) = I$
B: $\varepsilon_1\varepsilon_{n-1}\varepsilon_n = -1$	B.1: $\varepsilon_2 = 1, \varepsilon_{n-2}\varepsilon_n = 1$	$C(A)$ is not nonnegative
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This chart is for $n \geq 3$. The case $n = 2$ is studied in our reference: JAM-2014

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TP matrices $\subset$ A.1

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TP matrices $\subset$ A.1

t.n.p. matrices $\subset$ B.3

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Conclusions

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- We have established the signs conditions for 5 coordinates of the signature. Could we generalize these results to non-sign-regular matrices satisfying the 5-order sign conditions specified?

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Thanks for your attention

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