Characterizations of totally nonpositive and totally negative rectangular matrices:

Relationships with STP and TP matrices.

Rafael Cantó

Institut de Matemàtica Multidisciplinar
Universitat Politècnica de València. Spain.

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Joint work with Ana M. Urbano

Problem to study

$$A = (a_{ij}) \in \mathbb{R}^{n \times m} \text{ with rank } r$$

The most common problem studied for an square or rectangular t.n.p. (t.n.) matrix is:

Characterize A in terms of the factors of its LDU factorization using the Gaussian or the Neville elimination process with no pivoting.

$$\Downarrow$$

To reduce the number of minors to be checked in order to decide whether or not A is t.n.p. (t.n.)

t.n. (totally negative) $\Rightarrow$ all its minors are negative.

t.n.p. (totally nonpositive) $\Rightarrow$ all its minors are nonpositive.

Lower (upper) triangular ΔSTP $\Rightarrow$ all its nontrivial minors are positive.

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \quad \text{SI} \qquad L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix} \quad \text{NO}$$

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Classification

$$A = (a_{ij}) \in \mathbb{R}^{n \times m} \text{ with rank } r$$

Consider different cases:

- A is a t.n. matrix ($a_{ij} < 0$).

- A is a t.n.p. matrix ($a_{11} \leq 0$, $a_{nm} \leq 0$, $a_{ij} < 0$):

$\boxed{a_{11} < 0}$	$\boxed{a_{11} = 0}$
• $\boxed{A \text{ is nonsingular}}$	• A is nonsingular
• A is rectangular	• A is rectangular

A $n \times n$ is t.n. $(a_{ij} < 0)$

[Gasca-Peña, 1994]
They obtained a characterization in terms of the parameters computed from the Neville elimination.

[Fallat-Van Den Driessche, 2000]
They studied spectral properties and the *UDL* factorization.

M. Gasca, J.M. Peña, *A test for strict sign-regularity*, LAA 197/198, 133-142, 1994.
S.M. Fallat, P. Van Den Driessche, *On matrices with all minors negative*, ELA 7, 92-99, 2000.

A $n \times n$ is t.n. ($a_{ij} < 0$). Basic results

Using properties of strictly sign regular matrices and by Perron-Frobenius (since $-A$ is entrywise positive):

- All the eigenvalues of A are real and distinct.
- Exactly one eigenvalue is negative, and this eigenvalue is simple, has the largest modulus and has an eigenvector with all entries positive.

By continuity it follows that if A is an $n \times n$ t.n.p. matrix, then

- All the eigenvalues of A are real and at most one is negative.

$$A = \begin{bmatrix} -20 & -16 & -4 \\ -16 & -12.2 & -2.8 \\ -4 & -2.8 & -0.2 \end{bmatrix} \quad \text{is t.n.}$$

$$\sigma(A) = \{-33.29, 0.72, 0.17\} \quad v_1 = (0.78, 0.61, 0.15)$$

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A $n \times n$ is t.n. ($a_{ij} < 0$). Triangular factorization

Using properties of the *LDU* factorizations of STP and TP matrices:

Theorem 1

$A = (a_{ij}) \in \mathbb{R}^{n \times n}$. A is t.n. $\iff$ $A = LDU$

- L (U) is a unit lower (upper) triangular ΔSTP matrix
- $D = \mathrm{diag}(-d_1, d_2, d_3, \ldots, d_n)$ with $d_i > 0$, for $i = 1, 2, \ldots, n$.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 4/5 & 1 & 0 \\ 1/5 & 2/3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -20 & 0 & 0 \\ 0 & 3/5 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 4/5 & 1/5 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

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A $n \times m$ is t.n. Extension

We derive a characterization of t.n. rectangular matrices using their full rank factorization in echelon form. [C.-Ricarte-Urbano, 2009]
This result for rectangular STP matrices is given in [Gasca-Peña, 1992]

Theorem 2

$A = (a_{ij}) \in \mathbb{R}^{n\times m}$, $n \leq m$

$$A \text{ is STP}$$
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Full rank factorization in echelon form

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A $n \times m$ is t.n. Necessary condition

Theorem 3

$A = (a_{ij}) \in \mathbb{R}^{n \times m}$, $n \leq m$

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Sketch of the proof

Applying the first iteration of the Neville elimination process to A we have

$$E_{(1)}A = \left[\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1m} \\ \hline 0 & & & \\ \vdots & & \bar{A}_{22} & \\ 0 & & & \end{array}\right]$$

$$= \begin{bmatrix} a_{11} & O \\ O & I_{n-1} \end{bmatrix} \left[\begin{array}{c|ccc} 1 & a_{12}/a_{11} & \cdots & a_{1m}/a_{11} \\ \hline 0 & & & \\ \vdots & & \bar{A}_{22} & \\ 0 & & & \end{array}\right].$$

Since all nontrivial minors of

$$\bar{A} = \left[\begin{array}{c|ccc} 1 & a_{12}/a_{11} & \cdots & a_{1m}/a_{11} \\ \hline 0 & & & \\ \vdots & & \bar{A}_{22} & \\ 0 & & & \end{array}\right]$$ are positive, then

$\bar{A}_{22} \in \mathbb{R}^{(n-1)\times(m-1)}$ is an STP matrix and it admits the full rank decomposition in echelon form $\bar{A}_{22} = \bar{L}_{22}\bar{D}_{22}\bar{U}_{22}$, where $\bar{L}_{22} \in \mathbb{R}^{(n-1)\times(n-1)}$ is a unit lower ΔSTP matrix, $\bar{D}_{22}$ is a diagonal matrix with positive main diagonal and $\bar{U}_{22} \in \mathbb{R}^{(n-1)\times(m-1)}$ is a unit upper ΔSTP matrix.

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$$= \begin{bmatrix} a_{11} & O \\ O & I_{n-1} \end{bmatrix} \begin{bmatrix} 1 & O \\ O & \bar{L}_{22} \end{bmatrix} \begin{bmatrix} 1 & O \\ O & \bar{D}_{22} \end{bmatrix} \left[\begin{array}{c|ccc} 1 & a_{12}/a_{11} & \cdots & a_{1m}/a_{11} \\ \hline 0 & & & \\ \vdots & & \bar{U}_{22} & \\ 0 & & & \end{array}\right]$$

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Then $A = \left(\left(E_{(1)}\right)^{-1}\bar{L}\right)\bar{D}\bar{U} = LDU$, where it is easy to see that L is a unit lower triangular ΔSTP matrix, $D = \bar{D} = \mathrm{diag}(-d_1, d_2, \ldots, d_n)$ with $d_i > 0$ for $i = 1, 2, \ldots, n$, $\boxed{-d_1 = a_{11}}$ and $U = \bar{U}$ is a unit upper echelon ΔSTP matrix.

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The converse of Theorem 3 is not true in general

$A = LDU =$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 4 \end{bmatrix}}_{U}$$

$$= \begin{bmatrix} -3 & -3 & -6 & -9 \\ -6 & -5 & -9 & -12 \\ -6 & -3 & -2 & \boxed{4} \end{bmatrix}$$

L, D and U satisfy the conditions of Theorem 3, but

$\boxed{A \text{ is not t.n.}}$

Theorem 4 (Sufficient condition)

$A = (a_{ij}) = LDU \in \mathbb{R}^{n \times m}$, with $\boxed{a_{nm} < 0}$ and the same conditions for L, D, U.

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Characterization of t.n. rectangular matrices

Theorem 5

$A = (a_{ij}) \in \mathbb{R}^{n\times m}$, with $\boxed{a_{nm} < 0}$

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$$A = LDU$$

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- $D = \text{diag}(-d_1,\, d_2,\, d_3,\, \ldots,\, d_n)$ with $d_i > 0$, for $i = 1, 2, \ldots, n$.

Example

$$A = LDU = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & \boxed{2} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 4 \end{bmatrix}}_{U}$$

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A $n \times m$ is t.n.p. with $a_{11} < 0$ and rank r

Theorem 6

$A = (a_{ij}) \in \mathbb{R}^{n \times m}$ with $a_{11} < 0$, $\boxed{a_{nm} \leq 0}$ $a_{ij} < 0$, $\operatorname{rank}(A) = r \leq \min\{n, m\}$.

A is a t.n.p. matrix

$$\Updownarrow$$

$$A = LDU$$

Full rank factorization in echelon form

- $L \in \mathbb{R}^{n \times r}$ is a unit lower echelon TP matrix with $l_{ij} > 0$, $i > j$, i.e., all entries that are not trivially zero are positive
- $U \in \mathbb{R}^{r \times m}$ is a unit upper echelon TP matrix with $u_{ij} > 0$, $i < j$, i.e., all entries that are not trivially zero are positive
- $D = \operatorname{diag}(-d_1, d_2, \ldots, d_r)$ with $d_i > 0$, for $i = 1, 2, \ldots, r$
- $\operatorname{rank}(L) = \operatorname{rank}(D) = \operatorname{rank}(U) = r$

$$U = \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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- $\operatorname{rank}(L) = \operatorname{rank}(D) = \operatorname{rank}(U) = r$

$$U = \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

$A = LDU =$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 4 \end{bmatrix}}_{U}$$

$$= \begin{bmatrix} -3 & -3 & -6 & -9 \\ -6 & -5 & -9 & -12 \\ -6 & -4 & -5 & -2 \end{bmatrix}$$

A is t.n.

$$B = \begin{bmatrix} -3 & -3 & -6 & -9 \\ -6 & -5 & -9 & -12 \\ -9 & -8 & -15 & -21 \end{bmatrix}$$

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A $n \times n$ is t.n.p. with $a_{11} = 0$

A nonsingular. [C.-Koev-Ricarte-Urbano, 2008]
If $a_{nn} < 0$, then $A = UDL$ by permutation similarity.

If $a_{nn} = 0$, then we cannot use Gauss or Neville with no pivoting:

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & -\frac{1}{2} \\ -1 & -\frac{3}{4} & 0 \end{bmatrix}$$

[Huang-Chu, 2010] They obtain a characterization in terms of the sign of its minors with consecutive initial rows or consecutive initial columns.

R. Huang, D. Chu, *Total nonpositivity of nonsingular matrices*, LAA 432, 2931-2941, 2010.

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Objective

$A = (a_{ij}) \in \mathbb{R}^{n\times n}$ nonsingular t.n.p. matrix with $\boxed{a_{11} = 0}$

$\Downarrow$ [C.-Ricarte-Urbano, 2013]

A is characterized in terms of a quasi-$\tilde{L}DU$ factorization:

$$\boxed{A = \tilde{L}DU}$$

$\boxed{\tilde{L}}$ is a $\boxed{\text{block}}$ lower triangular matrix,
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A rectangular with arbitrary rank: We have studied the extension of the same result.

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Properties of nonsingular t.n.p. matrices with $a_{11} = 0$

Proposition 1 [Peña, laa03, Th. 2.1(i)]

$$A = (a_{ij}) \in \mathbb{R}^{n \times n} \text{ nonsingular t.n.p. matrix with } a_{11} = 0.$$

$$\Downarrow$$

$$a_{ij} < 0 \text{ for all } i, j = 1, 2, \ldots, n, \text{ with } (i, j) \neq (n, n).$$

Proposition 2 [Huang-Chu, laa10, Th.5]

$$A = (a_{ij}) \in \mathbb{R}^{n \times n} \text{ nonsingular t.n.p. matrix with } a_{11} = 0.$$

$$\Downarrow$$

$$\det A[1, 2, \ldots, k] < 0 \text{ for } k = 2, 3, \ldots, n.$$

Main idea: $a_{11} = 0 \Rightarrow A \neq LDU$ with no pivoting

$P = [2, 1, 3, \ldots, n]$ permutation matrix,

$$B = PA = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n-1} & a_{2n} \\ 0 & a_{12} & \cdots & a_{1n-1} & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n-1} & a_{3n} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-11} & a_{n-12} & \cdots & a_{n-1n-1} & a_{n-1n} \\ a_{n1} & a_{n2} & \cdots & a_{nn-1} & a_{nn} \end{bmatrix}.$$

Let $n \geq 3$, by Proposition 2 we can obtain $B = L_B D_B U_B$ using the Gauss elimination process with no pivoting.

Factors U_B, D_B and L_B satisfy the following properties:

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Properties of U_B

$$U_B = \begin{bmatrix} 1 & u_{12} & \cdots & u_{1n-1} & u_{1n} \\ 0 & 1 & \cdots & u_{2n-1} & u_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & u_{n-1n} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Proposition 3

U_B is a unit upper triangular TP matrix with all entries above the main diagonal positive, i.e.,

$$\boxed{u_{ij} > 0} \quad \text{for } i = 1, 2, \ldots, n-1, \text{ and } j = i+1, \ldots, n.$$

Properties of D_B

$$D_B = \begin{bmatrix} -d_1 & 0 & 0 & \cdots & 0 \\ 0 & -d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

Proposition 4

D_B has all its diagonal entries positive except for the $(1,1)$ and $(2,2)$ entries, which are negative, i.e.,

$$\boxed{d_i > 0} \quad \text{for } i = 1, 2, 3, \ldots, n.$$

Properties of L_B

$$L_B = \left[\begin{array}{cc|cccc} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \hline l_{31} & \boxed{l_{32}} & 1 & \cdots & 0 & 0 \\ l_{41} & \boxed{l_{42}} & l_{43} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ l_{n-11} & \boxed{l_{n-12}} & l_{n-13} & \cdots & 1 & 0 \\ l_{n1} & \boxed{l_{n2}} & l_{n3} & \cdots & l_{nn-1} & 1 \end{array}\right]$$

Proposition 5

- The entries of the first column $l_{i1} > 0, \ i = 3, 4, \ldots, n$.
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  - if $l_{32} = 0 \Rightarrow l_{i2} = 0$ for $i = 4, 5, \ldots, n$,
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Nonsingular t.n.p. matrices with $a_{11} = 0$

Theorem 7

$A = (a_{ij}) \in \mathbb{R}^{n \times n}$ nonsingular t.n.p. matrix with $a_{11} = 0$.

Then, A has a unique factorization, $A = PB = PL_B D_B U_B = \tilde{L}DU$ where

- U unit upper triangular TP matrix with $u_{ij} > 0$, $i < j$,
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- $\tilde{L}$ is the block lower triangular matrix

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad \text{with} \quad \tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

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The converse of Theorem 7 is not true in general

$A = \tilde{L}DU =$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & 1 \end{bmatrix}}_{\tilde{L}} \underbrace{\begin{bmatrix} -15 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{U}$$

$$= \begin{bmatrix} 0 & -2 & -4 & -6 \\ -15 & -15 & -15 & -15 \\ -15 & -13 & -10 & -5 \\ -30 & -24 & -14 & \boxed{6} \end{bmatrix}$$

$\tilde{L}$, D and U satisfy the conditions of Theorem 7, but

$\boxed{A \text{ is not t.n.p.}}$

Necessary and sufficient condition

Theorem 8

$A = (a_{ij}) \in \mathbb{R}^{n \times n}$ nonsingular with $a_{11} = 0$, $\boxed{a_{nn} \le 0}$, $a_{ij} < 0$.

$$\boxed{A \text{ is a t.n.p. matrix}}$$

$$\Updownarrow$$

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Conclusions when A is nonsingular

$$A = (a_{ij}) \in \mathbb{R}^{n\times n} \text{ nonsingular t.n.p. matrix.}$$
$$a_{11} = 0 \Rightarrow A \neq LDU \text{ with no pivoting}$$

(1) $B = PA, \quad P = [2, 1, 3, \ldots, n] \Rightarrow A = \tilde{L}DU$ where

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- D is a diagonal matrix, and
- U is a unit upper triangular TP matrix.

(2) $C = AP, \quad P = [2, 1, 3, \ldots, n] \Rightarrow A = L\bar{D}\tilde{U}$ where

- L is a unit lower triangular TP matrix,
- $\bar{D}$ is a diagonal matrix, and
- $\boxed{\tilde{U}}$ is a $\boxed{\text{block}}$ upper triangular matrix.

Example

$$A = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & -\frac{1}{2} \\ -1 & -\frac{3}{4} & 0 \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -\frac{1}{4} & 1 \end{bmatrix}}_{\tilde{L}} \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{3}{4} & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}}_{\bar{D}} \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}}_{\tilde{U}}$$

$\tilde{L}$, D and U (L, $\bar{D}$, $\tilde{U}$) satisfy the conditions of Theorem 8, then

A is t.n.p.

Example

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$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{3}{4} & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}}_{\bar{D}} \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}}_{\tilde{U}}$$

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Example

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$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{3}{4} & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}}_{\bar{D}} \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}}_{\tilde{U}}$$

$\tilde{L}$, D and U (L, $\bar{D}$, $\tilde{U}$) satisfy the conditions of Theorem 8, then

A is t.n.p.

A is rectangular with arbitrary rank

Theorem 9 [C.-Ricarte-Urbano, 2014]

$A = (a_{ij}) \in \mathbb{R}^{n\times m}$ with $a_{11} = 0,\ a_{nm} \le 0\ \ a_{ij} < 0$, $\mathrm{rank}(A) = r$

$\boxed{A \text{ is a t.n.p. matrix}} \iff A$ has a unique $\boxed{\text{full rank}}$ factorization, $A = \tilde{L}DU$ s.t.

- U unit upper $\boxed{\text{echelon } r\times m}$ TP matrix with $u_{ij} > 0,\ i < j$,
- $D = \mathrm{diag}(-d_1, -d_2, d_3, \ldots, d_r)$ with $d_i > 0$,
- $\tilde{L}$ is the block lower $\boxed{\text{echelon } n\times r}$ matrix
- $\boxed{\mathrm{rank}(\tilde{L}) = \mathrm{rank}(D) = \mathrm{rank}(U) = r}$

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad \text{with} \quad \tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$l_{i1} > 0,\ l_{i2} \le 0,\ \ i = 3, 4, \ldots, n,$
$\tilde{L}_{22}$ is a unit lower $\boxed{\text{echelon}}$ TP matrix with positive entries under the $\boxed{\text{leading entry in each column}}$ and $\forall \alpha \in \mathcal{Q}_{k,n}$

$$\det \tilde{L}[\alpha|1, 2, \ldots, k] \le 0, \qquad k = 2, 3, \ldots, r.$$

A is rectangular with arbitrary rank

Theorem 9 [C.-Ricarte-Urbano, 2014]

$A = (a_{ij}) \in \mathbb{R}^{n \times m}$ with $a_{11} = 0$, $a_{nm} \leq 0$ $a_{ij} < 0$, $\mathrm{rank}(A) = r$

$\boxed{A \text{ is a t.n.p. matrix}} \iff A$ has a unique $\boxed{\text{full rank}}$ factorization, $A = \tilde{L}DU$ s.t.

- U unit upper $\boxed{\text{echelon } r \times m}$ TP matrix with $u_{ij} > 0$, $i < j$,
- $D = \mathrm{diag}(-d_1, -d_2, d_3, \ldots, d_r)$ with $d_i > 0$,
- $\tilde{L}$ is the block lower $\boxed{\text{echelon } n \times r}$ matrix
- $\boxed{\mathrm{rank}(\tilde{L}) = \mathrm{rank}(D) = \mathrm{rank}(U) = r}$

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & O \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}, \quad \text{with} \quad \tilde{L}_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$l_{i1} > 0$, $l_{i2} \leq 0$, $i = 3, 4, \ldots, n$,
$\tilde{L}_{22}$ is a unit lower $\boxed{\text{echelon}}$ TP matrix with positive entries under the $\boxed{\text{leading entry in each column}}$ and $\forall \alpha \in \mathcal{Q}_{k,n}$

$$\det \tilde{L}[\alpha|1, 2, \ldots, k] \leq 0, \quad k = 2, 3, \ldots, r.$$

THANKS FOR YOUR ATTENTION

Sketch of the proof

$A = (a_{ij}) \in \mathbb{R}^{n\times m}$ with $a_{11} = 0$, consider:

- $n < m$
- A has nonzero rows and nonzero columns.

Cases:
(1) A has full row rank ($\text{rank}(A) = n$).
(2) A has arbitrary rank ($\text{rank}(A) = r < n$).

($\Longrightarrow$)

(1) Let $\bar{A} = A[1, 2, \ldots, n|1, 2, s_3, \ldots, s_n] \in \mathbb{R}^{n\times n}$ be the matrix formed by the first n linearly independent columns of A and use [C.-Ricarte-Urbano, laa12, Th. 1].

(2) Let $A_1 = A[1, 2, i_3, \ldots, i_r|1, 2, \ldots, m] \in \mathbb{R}^{r\times m}$ be the matrix formed by the first r linear independent rows of A and use (1).

Sketch of the sufficient condition

$(\Longleftarrow)$

(1) Consider $A = (a_{ij}) = \tilde{L}DU \in \mathbb{R}^{n\times m}$

Let $L = P\tilde{L} \in \mathbb{R}^{n\times n}$, where $P = [2,\ 1,\ 3,\ \dots,\ n]$, $D \in \mathbb{R}^{n\times n}$, $U \in \mathbb{R}^{n\times m}$, and $\boxed{B = PA = LDU} \in \mathbb{R}^{n\times m}$

$$\Downarrow$$

We construct the square matrices $L(\delta) \in \mathbb{R}^{m\times m}$, $D(\delta) \in \mathbb{R}^{m\times m}$ and $U(\delta) \in \mathbb{R}^{m\times m}$ s.t.

- the (m, m) entry of $B(\delta) = L(\delta)D(\delta)U(\delta)$ is nonpositive
- $B = B(\delta)[1, 2, \dots, n|1, 2, \dots, m]$.

$\Downarrow$ by [C.-Ricarte-Urbano, laa12, Th. 2]

$A(\delta) = PB(\delta)$ is t.n.p. and

$$\boxed{A = PB(\delta)[1, 2, \dots, n|1, 2, \dots, m] \text{ is also t.n.p.}}$$

Sketch of the sufficient condition (cont.)

$(\Longleftarrow)$

(2) Consider $A = (a_{ij}) = \tilde{L}DU \in \mathbb{R}^{n\times m}$, $\text{rank}(A) = r$.

$$\tilde{L}\ n \times r,\ D\ r \times r,\ U\ r \times m.$$

- We construct

$$\tilde{L}(\delta) = \begin{bmatrix} \tilde{L}(\delta)_{1_1} & 0 \\ \tilde{L}(\delta)_{1_2} & \tilde{L}(\delta)_{2_2} \end{bmatrix} \in \mathbb{R}^{n\times n}$$

$\tilde{L}(\delta)_{1_1}$ $r \times r$ with r L.I. rows.

- Similar for

$$U(\delta) = \begin{bmatrix} U(\delta)_{1_1} & U(\delta)_{1_2} \\ 0 & U(\delta)_{2_2} \end{bmatrix} \in \mathbb{R}^{n\times m}$$

$U(\delta)_{1_1}$ $r \times r$ with r L.I. rows.

- $D(\delta) = \text{diag}(-d_1, -d_2, d_3, \ldots, d_r, \delta, \delta, \ldots, \delta) \in \mathbb{R}^{n\times n}$

$$\text{Let } \boxed{A(\delta) = \tilde{L}(\delta)D(\delta)U(\delta)} \in \mathbb{R}^{n\times m}.$$

Sketch of the sufficient condition (cont.)

$$\boxed{A(\delta) = \tilde{L}(\delta)D(\delta)U(\delta)} \in \mathbb{R}^{n\times m}$$

$$\Downarrow$$

there exists δ_0 s.t. $A(\delta)(n,m) < 0$ for all $\delta < \delta_0$.

$$\Downarrow$$

$A(\delta)$ is t.n.p. for all $\delta < \delta_0$

$$\Downarrow$$

for all $\alpha \in \mathcal{Q}_{k,n},\ \ \beta \in \mathcal{Q}_{k,m},\ \ k = 1, 2, \ldots, n$

$$\det A[\alpha|\beta] = \lim_{\delta\to 0} \det A(\delta)[\alpha|\beta] \leq 0$$

$$\Downarrow$$

$$\boxed{A \text{ is t.n.p.}}$$

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