Numerical methods for the palindromic eigenvalue problem

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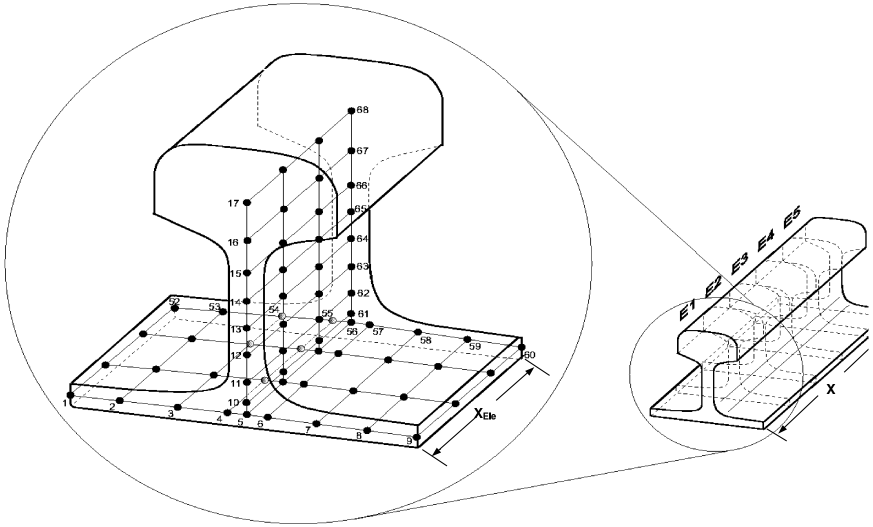
The application

Palindromic eigenvalue problems

Application: vibration analysis of rail tracks excited by high speed trains



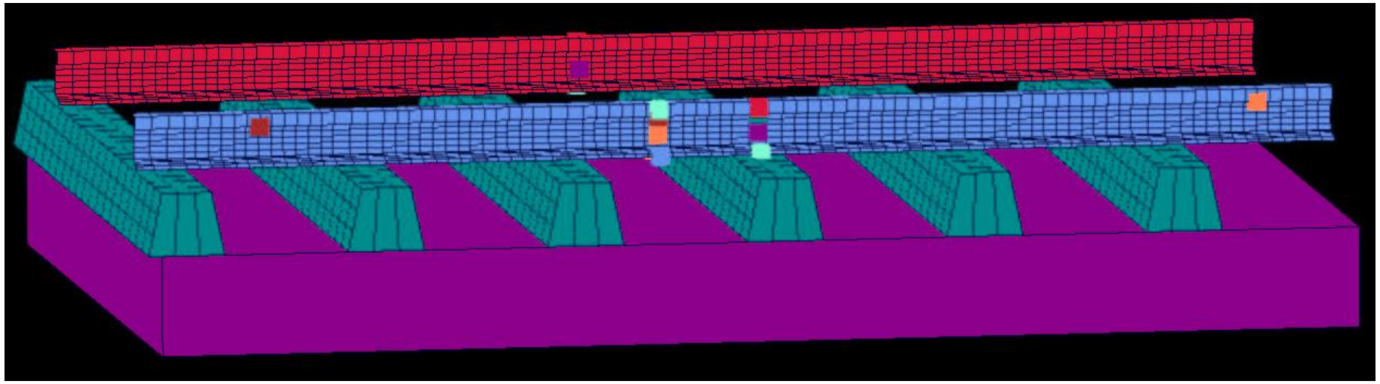
Finite element discretization of rail leads to the eigenvalue problem

$$\big(\lambda^2 A_1(\omega)^T + \lambda A_0(\omega) + A_1(\omega)\big) y = 0,$$

where $A_0, A_1 \in \mathbb{C}^{n\times n}$ and where ω is the excitation frequency.

Palindromic eigenvalue problems in applications

Example: vibration analysis of rail tracks under excitation arising from fast trains



Palindromic eigenvalue problems in applications

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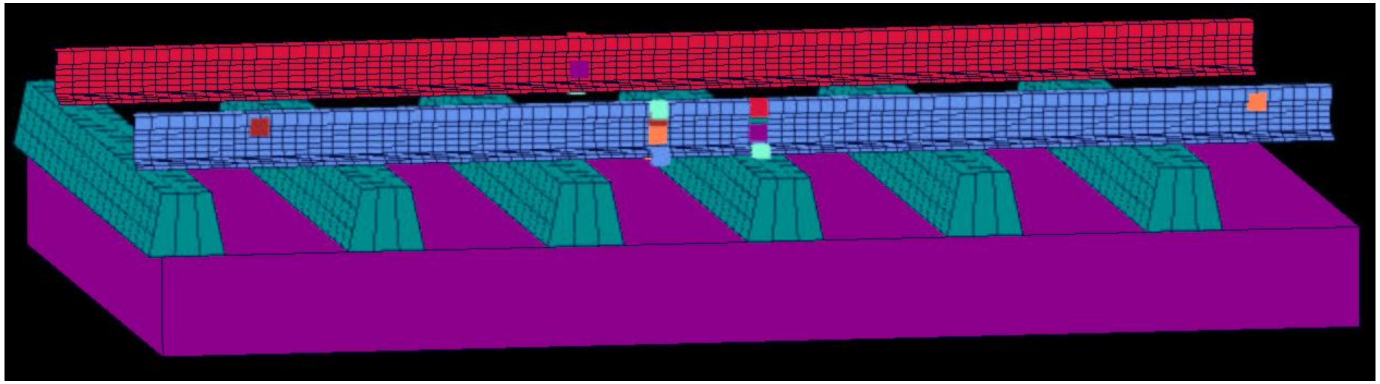

Assumptions:

1. the rail is infinite;

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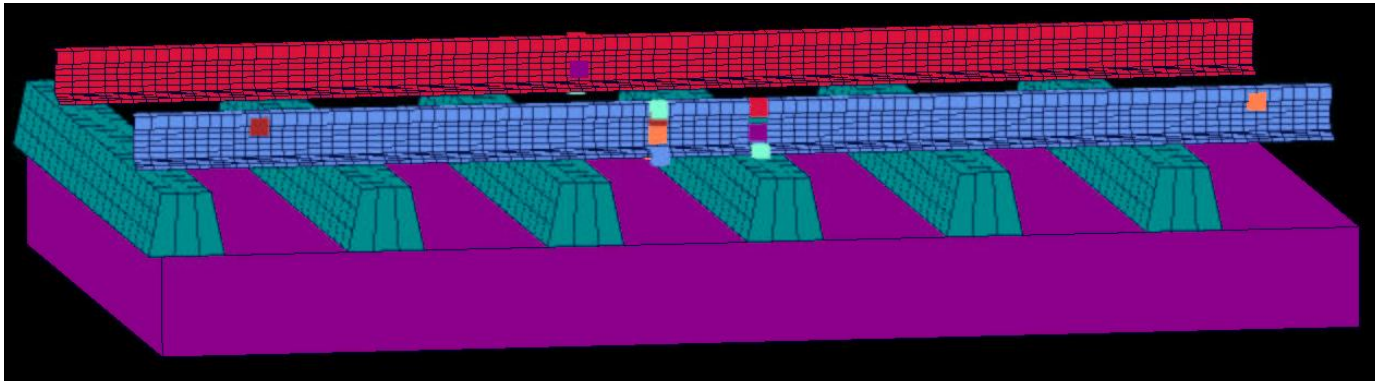
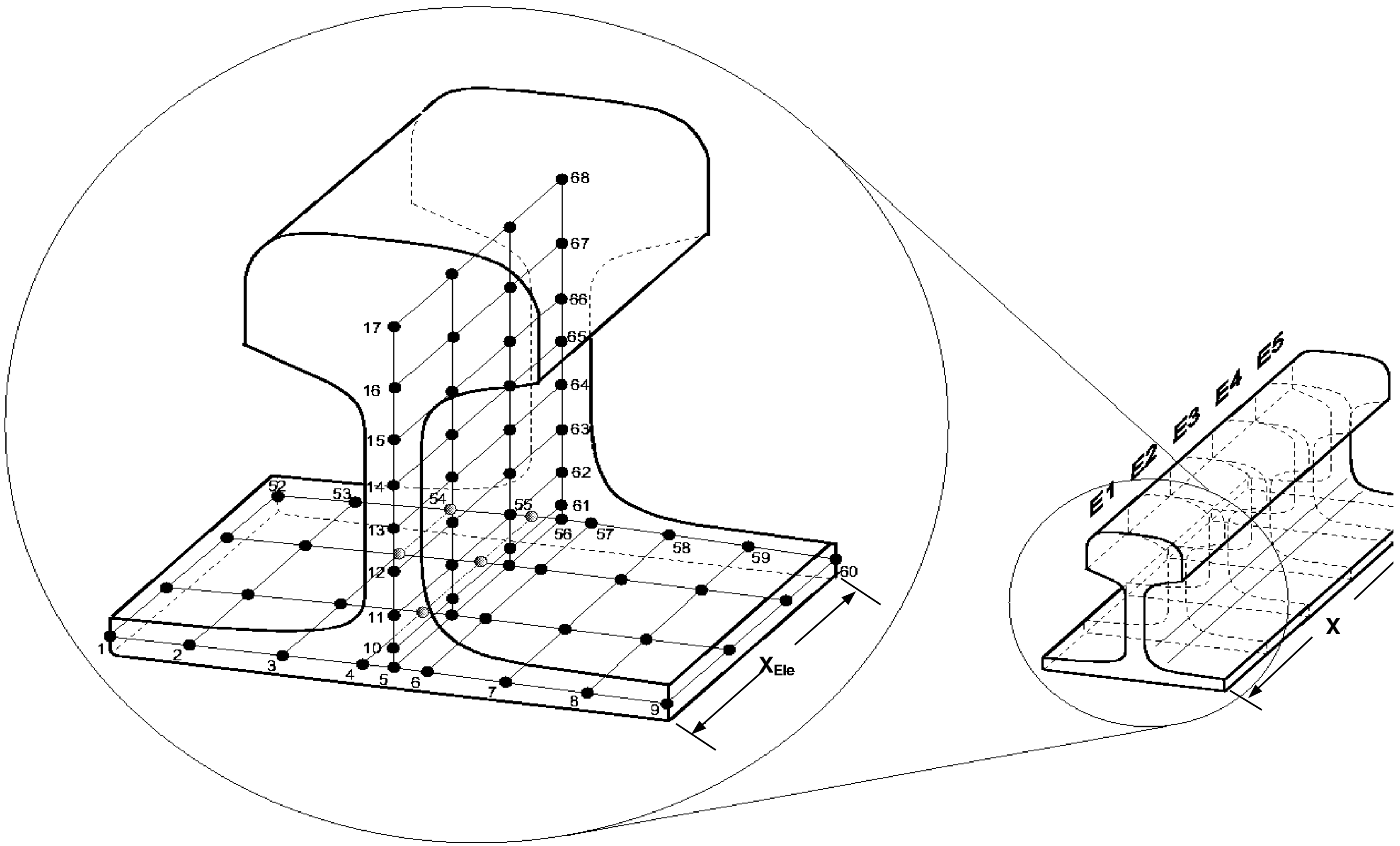


Assumptions:

1. the rail is infinite;
2. identical form of track between two ties $\Rightarrow$ the system is periodic;

FE discretization of the rail



FE discretization of the rail

This leads to an infinite second order system

$$M\ddot{x} + D\dot{x} + Sx = F,$$

where

$$M = \begin{bmatrix} \ddots & \ddots & 0 & \dots & 0 \\ \ddots & M_{j-1,0} & M_{j,1} & 0 & \dots \\ 0 & M_{j,1}^T & M_{j,0} & M_{j+1,1} & 0 \\ \vdots & \ddots & M_{j+1,1}^T & M_{j+1,0} & \ddots \\ 0 & \dots & 0 & \ddots & \ddots \end{bmatrix}, \quad x = \begin{bmatrix} \vdots \\ x_{j-1} \\ x_j \\ x_{j+1} \\ \vdots \end{bmatrix}, \quad F = \begin{bmatrix} \vdots \\ F_{j-1} \\ F_j \\ F_{j+1} \\ \vdots \end{bmatrix}$$

The matrices D and S have a similar structure.

A related difference equation

Ansatz: $F_j = \hat{F}_j e^{i\omega t}$, $x_j = \hat{x}_j e^{i\omega t}$, where ω is the excitation frequency

$$\rightsquigarrow A_{j-1,j}^T \hat{x}_{j-1} + A_{jj}\hat{x}_j + A_{j,j+1}\hat{x}_{j+1} = \hat{F}_j,$$

where

$$\begin{aligned} A_{j,j+1} &= -\omega^2 M_{j1} + i\omega D_{j1} + S_{j1} \\ A_{jj} &= -\omega^2 M_{j0} + i\omega D_{j0} + S_{j0} \end{aligned}$$

Observation: the rail track has identical form between two ties

$\rightsquigarrow$ the system matrices vary periodically

A related difference equation

Combine the parts of the rail between to ties into one vector:

$$y_j = \left[\begin{array}{cccc} \hat{x}_j^T & \hat{x}_{j+1}^T & \ldots & \hat{x}_{j+l}^T \end{array} \right]^T$$

$\rightsquigarrow$ constant coefficient difference equation:

$$A_1^T y_{j-1} + A_0 y_j + A_1 y_{j+1} = \tilde{F}_j,$$

where

$$A_0 = \begin{bmatrix} A_{j,j} & A_{j,j+1} & & 0 \\ A_{j,j+1}^T & A_{j+1,j+1} & \ddots & \\ & \ddots & \ddots & A_{j+l-1,j+l} \\ 0 & & A_{j+l-1,j+l}^T & A_{j+l,j+l} \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 \\ A_{j+l,j+l+1} & 0 & \ldots & 0 \end{bmatrix}.$$

A quadratic eigenvalue problem

Ansatz: $y_{j+1} = \lambda y_j \quad \rightsquigarrow (A_1^T + \lambda A_0 + \lambda^2 A_1)y = 0$

- A_0 is complex symmetric ($A_0 = A_0^T$)
- A_1 is complex and singular
- coefficient sequence A_1^T–A_0–A_1 can also be read backwards

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- A_0 is complex symmetric ($A_0 = A_0^T$)
- A_1 is complex and singular
- coefficient sequence A_1^T–A_0–A_1 can also be read backwards
- This is a quadratic **palindromic eigenvalue problem**!

Palindromic matrix polynomials

Reminder: A matrix polynomial $P(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^k A_k$ is called T-palindromic (in short: **palindromic**) if

$$P(\lambda) = \sum_{j=0}^{k} \lambda^{k-j} A_j^T.$$

Examples:

- $P(\lambda) = A + \lambda B + \lambda^2 B^T + \lambda^3 A^T$;
- $P(\lambda) = A_2^T + \lambda A_1^T + \lambda^2 A_0 + \lambda^3 A_1 + \lambda^4 A_2$, where A_0 is symmetric;
- palindromic pencils $\lambda Z + Z^T$.

Formal resemblance with linguistic palindroms like "**¡La moral, claro, mal!**".

Properties of palindromic matrix polynomials

General assumption: all matrix polynomials under consideration are regular, i.e., $\det P(\lambda) \not\equiv 0$.

Spectral symmetry: Palindromic matrix polynomials have a **symplectic spectrum**.

- if λ_0 is an eigenvalue of $P(\lambda)$, then so is λ_0^{-1};
- pairing occurs also in algebraic, geometric, and partial multiplicities;
- symmetry degenerates for $\lambda_0 = 1$ and $\lambda_0 = -1$;

The linearization

How to solve structured eigenvalue problems

Given: structured eigenvalue problem $(\lambda^m A_m + \cdots + \lambda A_1 + A_0)x = 0.$

Standard approach: linearize the problem; use the companion form

$$\lambda \begin{bmatrix} A_m & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix} \begin{bmatrix} \lambda^{m-1}x \\ \vdots \\ \lambda x \\ x \end{bmatrix} = \begin{bmatrix} -A_{m-1} & \dots & \dots & -A_0 \\ I & & & \\ & \ddots & & \\ & & I & 0 \end{bmatrix} \begin{bmatrix} \lambda^{m-1}x \\ \vdots \\ \lambda x \\ x \end{bmatrix}$$

Task: find a linearization having the same symmetry structure as the original matrix polynomial that still has the nice properties of the companion form

An ansatz space for linearizations

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$$\lambda \underbrace{\begin{bmatrix} A_m & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{bmatrix} \lambda^{m-1} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}}_{=:\Lambda} x = \underbrace{\begin{bmatrix} -A_{m-1} & \dots & -A_1 & -A_0 \\ I & & & \\ & \ddots & & \\ & & I & 0 \end{bmatrix}}_{\mathcal{B}} \underbrace{\begin{bmatrix} \lambda^{m-1} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}}_{=\Lambda} x$$

The companion form $C(\lambda) = \lambda\mathcal{A} - \mathcal{B}$ satisfies

$$C(\lambda)\Lambda x = \begin{bmatrix} P(\lambda) \\ 0 \\ \vdots \\ 0 \end{bmatrix} x = e_1 \otimes P(\lambda)x$$

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Idea: Allow linearizations $L(\lambda)$ satisfying

$$L(\lambda)\Lambda x = \begin{bmatrix} v_1 P(\lambda) \\ v_2 P(\lambda) \\ \vdots \\ v_m P(\lambda) \end{bmatrix} x = v \otimes P(\lambda)x$$

Then $L(\lambda)$ still has the eigenvector recovery property as $C(\lambda)$.

We will call v an **ansatz vector**.

An ansatz space for linearizations

This can be said differently. Consider again $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{A} = \begin{bmatrix} A_m & 0 & \cdots & 0 \\ 0 & I & & \\ \vdots & & \ddots & \\ 0 & & & I \end{bmatrix}, \quad -\mathcal{B} = \begin{bmatrix} A_{m-1} & \dots & A_1 & A_0 \\ -I & & & 0 \\ & \ddots & & \vdots \\ & & -I & 0 \end{bmatrix}$$

Introduce the so-called **shifted sum**:

$$\begin{matrix} & n & n(m-1) \\ nm & [\,\mathcal{A}_1 & \mathcal{A}_2 \,] \end{matrix} \boxplus \begin{matrix} n(m-1) & n \\ [\,\mathcal{B}_1 & \mathcal{B}_2\,] \end{matrix} := \begin{matrix} n & n(m-1) & n \\ [\,\mathcal{A}_1 & \mathcal{A}_2 + \mathcal{B}_1 & \mathcal{B}_2\,] \end{matrix}$$

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We obtain:

$$\mathcal{A} \boxplus (-\mathcal{B}) = \begin{bmatrix} A_m & A_{m-1} & \dots & A_1 & A_0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

The first block row consists of the coefficients of the polynomial $P(\lambda)$.

An ansatz space for linearizations

Ansatz: Look for linearizations $L(\lambda)$ in the vector space

$$\mathbb{L}_1(P) = \left\{\lambda\mathcal{A} - \mathcal{B} \;:\; (\lambda\mathcal{A} - \mathcal{B})\Lambda = v \otimes P(\lambda), v = (v_1, \ldots, v_m)^T \in \mathbb{C}^m\right\}$$

$$= \left\{\lambda\mathcal{A} - \mathcal{B} \;:\; \mathcal{A} \boxplus (-\mathcal{B}) = \begin{bmatrix} v_1 A_m & \cdots & v_1 A_1 & v_1 A_0 \\ \vdots & \ldots & \vdots & \vdots \\ v_m A_m & \cdots & v_m A_1 & v_m A_0 \end{bmatrix}, v_i \in \mathbb{C}\right\}$$

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Theorem (Mackey, Mackey, M., Mehrmann, 2004):
$L(\lambda) \in \mathbb{L}_1(P)$ is a linearization of the regular matrix polynomial $P(\lambda)$ if and only if $L(\lambda)$ is regular, i.e., $\det L(\lambda) \not\equiv 0$.

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Note: These linearizations share nice properties with the companion form:

- eigenvectors have the form $\Lambda = \begin{bmatrix} \lambda^{m-1} x \\ \vdots \\ x \end{bmatrix}$, where $P(\lambda)x = 0$;
- information on eigenvalue ∞ is properly reflected (strong linearization)

Palindromic linearizations

Question: Can we find a palindromic $L(\lambda) \in \mathbb{L}_1(P)$?

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Answer: Yes! Use a *palindromic* v, i.e., $v \in \mathbb{R}^m$ satisfying

$$v = \left[\, v_1 \;\; v_2 \;\; \ldots \;\; v_{m-1} \;\; v_m \,\right] = \left[\, v_m \;\; v_{m-1} \;\; \ldots \;\; v_2 \;\; v_1 \,\right].$$

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Theorem (Mackey, Mackey, M., Mehrmann, 2004):
Let $v \in \mathbb{R}^m$ be palindromic. Then there exists a unique palindromic pencil $L(\lambda) = \lambda Z + Z^T$ such that

$$Z \boxplus Z^T = \begin{bmatrix} v_1 A_m & \cdots & v_1 A_1 & v_1 A_0 \\ \vdots & \dots & \vdots & \vdots \\ v_m A_m & \cdots & v_m A_1 & v_m A_0 \end{bmatrix} \in \mathbb{L}$$

Construction of palindromic linearizations

Given: Palindromic matrix polynomial $P(\lambda) = \lambda^3 A + \lambda^2 B + \lambda B^T + A^T$.
We construct a palindromic pencil corresponding to $v = (1, 0, 1)^T \in \mathbb{R}^3$.
We know:

$$Z \boxplus Z^T = \begin{bmatrix} A & B & B^T & A^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A & B & B^T & A^T \end{bmatrix}$$

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We obtain:

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This is a palindromic linearization of $P(\lambda)$ if $\det(\lambda Z + Z^T) \not\equiv 0$.

Question: How do we check regularity of $\lambda Z + Z^T \in \mathbb{L}_1(P)$?

The "v-polynomial theorem"

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Let $v = (v_1, \ldots, v_m)$ be palindromic and let $\lambda Z - Z^T \in \mathbb{L}_1(P)$ be associated with v. Consider the *v-polynomial* $\lambda^{m-1} v_1 + \cdots + \lambda v_{m-1} + v_m$. Then $\lambda Z - Z^T$ is regular if and only if the roots of the v-polynomial are disjoint from the eigenvalues of $P(\lambda)$.

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Example: The v-polynomial of $v = (1, 0, 1)^T \in \mathbb{R}^3$ is $\lambda^2 + 1$. Thus

$$\lambda \begin{bmatrix} A & B - A^T & B^T \\ 0 & A - B^T & B - A^T \\ A & 0 & A \end{bmatrix} + \begin{bmatrix} A^T & 0 & A^T \\ B^T - A & A^T - B & 0 \\ B & B^T - A & A^T \end{bmatrix}$$

is a linearization of $P(\lambda) = \lambda^3 A + \lambda^2 B + \lambda B^T + A^T$ if and only if $\pm i$ are no eigenvalues of $P(\lambda)$.

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Aim: Find a palindromic linearization for $P(\lambda) = \lambda^2 A_1 + \lambda A_0 + A_1^T$, where A_0 is complex symmetric.

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$$\lambda \begin{bmatrix} A_1 & A_0 - A_1^T \\ A_1 & A_1 \end{bmatrix} + \begin{bmatrix} A_1^T & A_1^T \\ A_0 - A_1 & A_1^T \end{bmatrix}$$

The railway eigenvalue problem

Aim: Find a palindromic linearization for $P(\lambda) = \lambda^2 A_1 + \lambda A_0 + A_1^T$, where A_0 is complex symmetric.

- There is only one palindromic $v \in \mathbb{R}^2$ up to scaling: $v = (1, 1)$.
- The corresponding v-polynomial is $\lambda + 1$.
- We obtain the following possible linearization:

$$\lambda \begin{bmatrix} A_1 & A_0 - A_1^T \\ A_1 & A_1 \end{bmatrix} + \begin{bmatrix} A_1^T & A_1^T \\ A_0 - A_1 & A_1^T \end{bmatrix}$$

- This is a linearization if and only if -1 is not an eigenvalue of $P(\lambda)$, i.e., if and only if $A_1 - A_0 + A_1^T$ is nonsingular.

The anti-triangular Schur form

How to solve linear palindromic eigenvalue problems

Task: Solve the generalized eigenvalue problem for $\lambda Z + Z^T$.

- T-congruence transformations preserve the structure:

$$(\lambda Z + Z^T) \mapsto P^T(\lambda Z + Z^T)P, \quad P \text{ invertible}$$

- Numerical stability: Choose $P = U$ unitary if possible.

- Look for condensed forms under simultaneous unitary consimilarity:

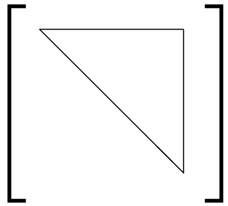
$$(\lambda Z + Z^T) \mapsto \overline{U}^{-1}(\lambda Z + Z^T)U, \quad U \text{ unitary}$$

Advantage: We have to store and work on Z only.

Anti-triangular forms

Naive approach: reduce Z to triangular form:

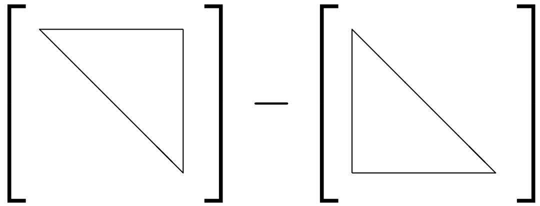
$$U^T Z U \quad \hat{=}$$



Anti-triangular forms

Naive approach: reduce Z to triangular form:

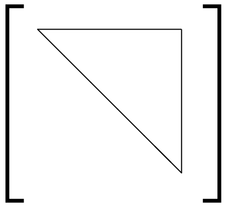
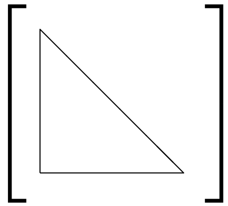
$$\lambda U^T Z U - U^T Z^T U \hat{=} \lambda$$



Drawback: the eigenvalues of $\lambda Z + Z^T$ cannot be read off;

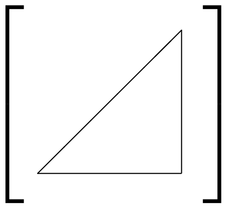
Anti-triangular forms

Naive approach: reduce Z to triangular form:

$$\lambda U^T Z U - U^T Z^T U \mathrel{\hat{=}} \lambda$$  $-$ 

Drawback: the eigenvalues of $\lambda Z + Z^T$ cannot be read off;

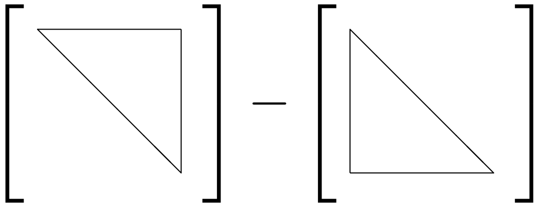
Idea: reduce Z to **anti-triangular form**;

$$U^T Z U \quad \mathrel{\hat{=}}$$ 

Anti-triangular forms

Naive approach: reduce Z to triangular form:

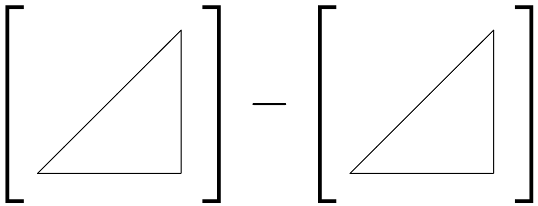
$$\lambda U^T Z U - U^T Z^T U \hat{=} \lambda$$



Drawback: the eigenvalues of $\lambda Z + Z^T$ cannot be read off;

Idea: reduce Z to **anti-triangular form**;

$$\lambda U^T Z U - U^T Z^T U \hat{=} \lambda$$



Then the eigenvalues of $\lambda Z + Z^T$ can be easily read off.

The anti-triangular Schur form

Theorem: Let $Z \in \mathbb{C}^{n\times n}$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$U^T Z U = \begin{bmatrix} 0 & \dots & 0 & z_{1n} \\ \vdots & \iddots & z_{2,n-1} & \vdots \\ 0 & \iddots & \iddots & \vdots \\ z_{n1} & \dots & \dots & z_{nn} \end{bmatrix}$$

is in **anti-triangular Schur form**.

We now have

$$\lambda U^T Z U + U^T Z U = \lambda \begin{bmatrix} 0 & \dots & 0 & z_{1n} \\ \vdots & \iddots & z_{2,n-1} & \vdots \\ 0 & \iddots & \iddots & \vdots \\ z_{n1} & \dots & \dots & z_{nn} \end{bmatrix} + \begin{bmatrix} 0 & \dots & 0 & z_{n1} \\ \vdots & \iddots & z_{n-1,2} & \vdots \\ 0 & \iddots & \iddots & \vdots \\ z_{1n} & \dots & \dots & z_{nn} \end{bmatrix}$$

The anti-triangular Schur form

Theorem: Let $Z \in \mathbb{C}^{n\times n}$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$U^T Z U = \begin{bmatrix} 0 & \dots & 0 & z_{1n} \\ \vdots & \iddots & z_{2,n-1} & \vdots \\ 0 & \iddots & \iddots & \vdots \\ z_{n1} & \dots & \dots & z_{nn} \end{bmatrix}$$

is in **anti-triangular Schur form**.

Consequence: If $\det(\lambda Z + Z^T) \not\equiv 0$ then the eigenvalues of $\lambda Z + Z^T$ are

$$-\frac{z_{n1}}{z_{1n}}, \dots, -\frac{z_{1n}}{z_{n1}}, \qquad \text{(where } \frac{z}{0} := \infty\text{)}.$$

The anti-triangular Schur form

Theorem: Let $Z \in \mathbb{C}^{n\times n}$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$U^T Z U = \begin{bmatrix} 0 & \dots & 0 & z_{1n} \\ \vdots & \iddots & z_{2,n-1} & \vdots \\ 0 & \iddots & \iddots & \vdots \\ z_{n1} & \dots & \dots & z_{nn} \end{bmatrix}$$

is in **anti-triangular Schur form**.

Observation: If x is an eigenvector of $\lambda Z + Z^T$, then:

$$\lambda x^T Z x = -x^T Z^T x = -x^T Z x$$

Consequence: $x^T Z x = 0$, i.e., x is Z-**neutral**, provided that $\lambda \neq -1$.

The anti-triangular Schur form

Proof (for the generic case $\lambda Z + Z^T$ is regular and -1 is not an eigenvalue):

$$Z = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$

- If x is a normalized eigenvector,
- then x is Z-neutral, i.e. $x^T Z x = 0$, i.e., x and $\overline{Zx}$ are orthogonal.
- Let $Q = [x, q_2, \ldots, q_n]$ be unitary with $q_n = \frac{\overline{Zx}}{\|Zx\|}$ (or $\frac{\overline{Z^T x}}{\|Z^T x\|}$, when $Zx = 0$).

The anti-triangular Schur form

Proof (for the generic case $\lambda Z + Z^T$ is regular and -1 is not an eigenvalue):

$$Z = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$

- If x is a normalized eigenvector,
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- Compute $\widetilde{Z} = Q^T Z Q$.

The anti-triangular Schur form

Proof (for the generic case $\lambda Z + Z^T$ is regular and -1 is not an eigenvalue):

$$\widetilde{Z} = \begin{bmatrix} 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$

- If x is a normalized eigenvector,
- then x is Z-neutral, i.e. $x^T Z x = 0$, i.e., x and $\overline{Zx}$ are orthogonal.
- Let $Q = [x, q_2, \ldots, q_n]$ be unitary with $q_n = \frac{\overline{Zx}}{\|Zx\|}$ (or $\frac{\overline{Z^T x}}{\|Z^T x\|}$, when $Zx = 0$).
- Compute $\widetilde{Z} = Q^T Z Q$. **Then** $q_j^T Z q_1 = q_j^T Z x = 0$ **for** $j = 1, \ldots, n-1$.

The anti-triangular Schur form

Proof (for the generic case $\lambda Z + Z^T$ is regular and -1 is not an eigenvalue):

$$\widetilde{Z} = \begin{bmatrix} 0 & 0 & 0 & 0 & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$

- If x is a normalized eigenvector,
- then x is Z-neutral, i.e. $x^T Z x = 0$, i.e., x and $\overline{Zx}$ are orthogonal.
- Let $Q = [x, q_2, \ldots, q_n]$ be unitary with $q_n = \frac{\overline{Zx}}{\|Zx\|}$ (or $\frac{\overline{Z^T x}}{\|Z^T x\|}$, when $Zx = 0$).
- Compute $\widetilde{Z} = Q^T Z Q$. **We also have $q_j^T Z^T q_1 = q_j^T Z^T x = 0, j = 1, \ldots, n-1$, because Zx and $Z^T x$ are linearly dependent (x is eigenvector).**

The anti-triangular Schur form

Proof (for the generic case $\lambda Z + Z^T$ is regular and -1 is not an eigenvalue):

$$\widetilde{Z} = \begin{bmatrix} 0 & 0 & 0 & 0 & * \\ 0 & \color{red}{*} & \color{red}{*} & \color{red}{*} & * \\ 0 & \color{red}{*} & \color{red}{*} & \color{red}{*} & * \\ 0 & \color{red}{*} & \color{red}{*} & \color{red}{*} & * \\ * & * & * & * & * \end{bmatrix}$$

- If x is a normalized eigenvector,
- then x is Z-neutral, i.e. $x^T Z x = 0$, i.e., x and $\overline{Zx}$ are orthogonal.
- Let $Q = [x, q_2, \ldots, q_n]$ be unitary with $q_n = \frac{\overline{Zx}}{\|Zx\|}$ (or $\frac{\overline{Z^T x}}{\|Z^T x\|}$, when $Zx = 0$).
- Compute $\widetilde{Z} = Q^T Z Q$. **Continue with the red submatrix**.

Method No. 1: Structured Deflation

Method 1: Structured Deflation

Reminder: construction of anti-triangular Schur form

- start: choose an eigenvector x of $\lambda Z + Z^T$.

- x is Z-neutral, i.e. $x^T Z x = 0$,
- if the corresponding eigenvalue is not -1.

- This allows the construction of a unitary U with x and $\overline{Zx}$ (after normalization) as columns.
- Congruence trafo with U gets Z "closer" to anti-triangular Schur form.

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Method 1: Structured Deflation

Idea: replace x with a suitable deflating subspace

- start: choose an eigenvector x of $\lambda Z + Z^T$.
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Method 1: Structured Deflation

Idea: replace x with a suitable deflating subspace

- start: let Q span a deflating subspace of $\lambda Z + Z^T$.

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- if the corresponding eigenvalue is not -1.

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Method 1: Structured Deflation

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- start: let Q span a deflating subspace of $\lambda Z + Z^T$.
- Q is Z-neutral, i.e. $Q^T Z Q = 0$,
- if the set Λ of associated eigenvalues is reciprocal-free:

$$\mu \in \Lambda \Rightarrow \frac{1}{\mu} \notin \Lambda;$$

- This allows the construction of a unitary U with x and $\overline{Zx}$ (after normalization) as columns.
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- This allows the construction of a unitary matrix U with Q and $\overline{ZQ}$ as "columns" (after orthogonalization and normalization).
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Method 1: Structured Deflation

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- Q is Z-neutral, i.e. $Q^T Z Q = 0$,
- if the set Λ of associated eigenvalues is reciprocal-free:

$$\mu \in \Lambda \Rightarrow \frac{1}{\mu} \notin \Lambda;$$

- This allows the construction of a unitary matrix U with Q and $\overline{ZQ}$ as "columns" (after orthogonalization and normalization).
- Congruence trafo with U gets Z to anti-triangular Schur form, if the deflating subspace has dimension $\frac{n}{2}$ or $\frac{n-1}{2}$.

Method 1: Structured Deflation

Theorem: (generalizes a trick by A. Laub for the computation of the Hamiltonian Schur form) Let $\lambda Z + Z^T \in \mathbb{C}^{2n\times 2n}$ be regular and let

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \left(\lambda \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix} + \begin{bmatrix} Y_{11} & Y_{12} \\ 0 & Y_{22} \end{bmatrix} \right) \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}.$$

be its generalized Schur decomposition, where $X_{11}, Y_{11} \in \mathbb{C}^{n\times n}$. If

$$\mu \in \sigma(\lambda X_{11} + Y_{11}) \implies \frac{1}{\mu} \notin \sigma(\lambda X_{11} + Y_{11})$$

then

$$U = \begin{bmatrix} W_{11} & Q_{11}^T R_n \\ W_{21} & Q_{12}^T R_n \end{bmatrix}, \qquad \left(R_n := \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} \right)$$

is unitary and

$$U^T Z U = \begin{bmatrix} 0 & Y_{11}^T R_n \\ R_n X_{11} & * \end{bmatrix}.$$

is in anti-triangular form.

Method 1: Structured Deflation

Algorithm: (for regular $\lambda Z + Z^T$ not having eigenvalues with modulus 1)

1. Compute the generalized Schur decomposition
$$\lambda Z + Z^T = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \left(\lambda \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix} + \begin{bmatrix} Y_{11} & Y_{12} \\ 0 & Y_{22} \end{bmatrix} \right) \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}.$$
2. Reorder the eigenvalues such that $\lambda X_{11} + Y_{11}$ contains all eigenvalues with $|\lambda| > 1$.
3. Set $U = \begin{bmatrix} W_{11} & Q_{11}^T R_n \\ W_{21} & Q_{12}^T R_n \end{bmatrix}$.
4. Compute $Z_{22} = \begin{bmatrix} R_n Q_{11} & R_n Q_{12} \end{bmatrix} Z \begin{bmatrix} Q_{11}^T R_n \\ Q_{12}^T R_n \end{bmatrix}$.
5. Set $\widetilde{Z} := \begin{bmatrix} 0 & Y_{11}^T R_n \\ R_n X_{11} & Z_{22} \end{bmatrix}$.

Method 1: Structured Deflation

Properties:

+ cost is essentially the cost of QZ with reordering;

− only applicable if Z has even dimension and if $\lambda Z + Z^T$ does not have eigenvalues with modulus 1;

− problems if there are eigenvalues with modulus close to ± 1; $\leadsto$ QZ might detect more or less than n eigenvalues λ with $|\lambda| > 1$.

Questions: Are there other methods?

Method No. 2: A Jacobi-like Method

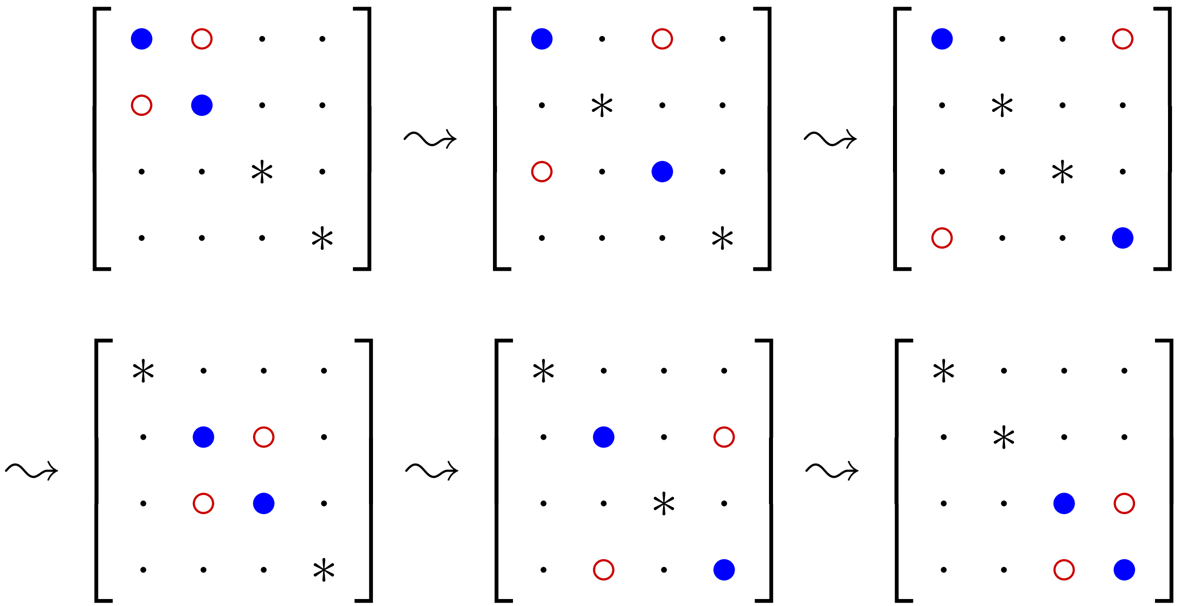
Jacobi(-like) methods

- 1846 Jacobi (symmetric matrices)
- ...
- 1955 Greenstadt (general complex matrices)
- 1985 Stewart (general complex matrices)
- 1990 Byers (Hamiltonian matrices)
- 1997 Bunse-Gerstner/Faßbender (Hamiltonian matrices)
- 2004 Hilliges, M., Mehrmann (palindromic pencils

Jacobi's method for symmetric matrices

Main theme: Elimination of one (or two) pivot elements by diagonalization of a 2×2 problem.

Sweep (cyclic-by-column): $n(n-1)/2$ steps with column-wise choise of pivots from top-to-bottom



Jacobi's method for symmetric matrices

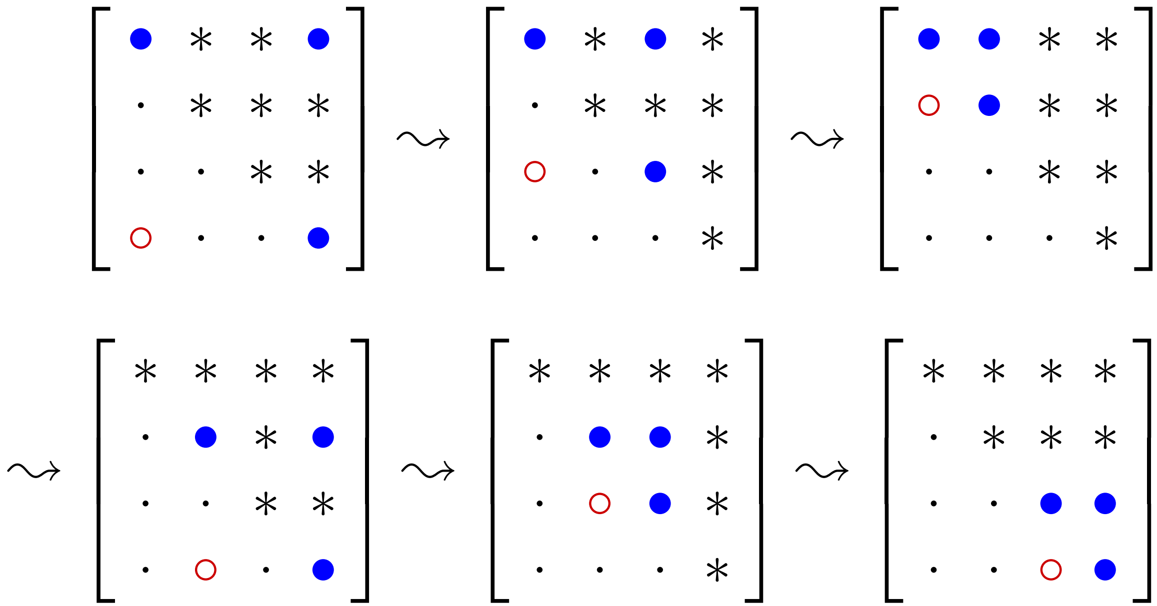
Properties:

- „off-Norm" off$(A) = \sqrt{\sum_{j \neq i} a_{ij}^2}$ is reduced in each step
- global convergence
- asymptotic quadratic convergence
- stable, parallelizable, more accurate than QR, fast if matrix is close to diagonal
- in general much slower than QR (not competitive)

Nonsymmetric Jacobi method

Main theme: Elimination of one (or two) pivot elements by trigonalization of a 2×2 problem.

Sweep (cyclic-by-column): $n(n-1)/2$ steps with column-wise choise of pivots from bottom-to-top

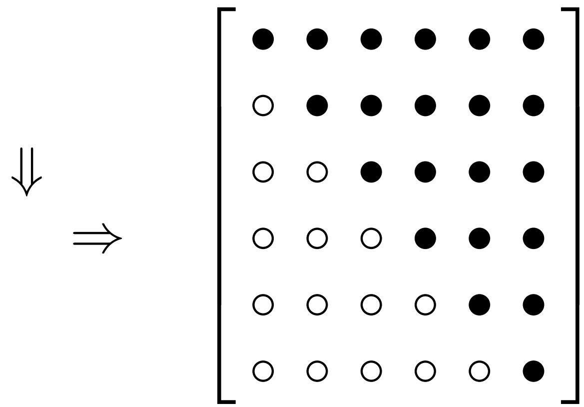


Surprise: sweep $\neq$ sweep

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet & \bullet & \bullet \\ \circ & \circ & \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \circ & \circ & \bullet \end{bmatrix}$$

Consider two “cyclic-by-column” sweeps:

Surprise: sweep $\neq$ sweep



Consider two "cyclic-by-column" sweeps:

i) "top-to-bottom": $(2,1),\ldots,(n,1),(3,2),\ldots,(n,2),\ldots,(n,n-1)$

Surprise: sweep $\neq$ sweep

$$\Uparrow \quad \Rightarrow \quad \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet & \bullet & \bullet \\ \circ & \circ & \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \circ & \circ & \bullet \end{bmatrix}$$

Consider two "cyclic-by-column" sweeps:

i) "top-to-bottom": $(2,1), \ldots, (n,1), (3,2), \ldots, (n,2), \ldots, (n, n-1)$

ii) "bottom-to-top": $(n,1), \ldots, (2,1), (n,2), \ldots, (3,2), \ldots, (n, n-1)$

Surprise: sweep $\neq$ sweep

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet & \bullet & \bullet \\ \circ & \circ & \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \circ & \circ & \bullet \end{bmatrix}$$

Consider two “cyclic-by-column” sweeps:

i) “top-to-bottom”: $(2,1),\ldots,(n,1),(3,2),\ldots,(n,2),\ldots,(n,n-1)$

ii) “bottom-to-top”: $(n,1),\ldots,(2,1),(n,2),\ldots,(3,2),\ldots,(n,n-1)$

symmetric case:
quadratic asymptotic convergence for both sweep selections;

Surprise: sweep $\neq$ sweep

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet & \bullet & \bullet \\ \circ & \circ & \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \circ & \circ & \bullet \end{bmatrix}$$

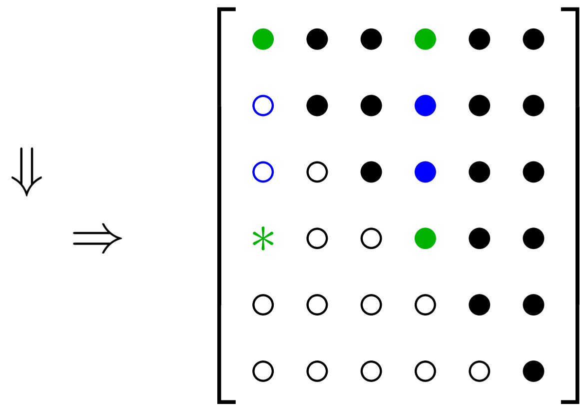
Consider two “cyclic-by-column” sweeps:

i) “top-to-bottom”: $(2,1),\ldots,(n,1),(3,2),\ldots,(n,2),\ldots,(n,n-1)$

ii) “bottom-to-top”: $(n,1),\ldots,(2,1),(n,2),\ldots,(3,2),\ldots,(n,n-1)$

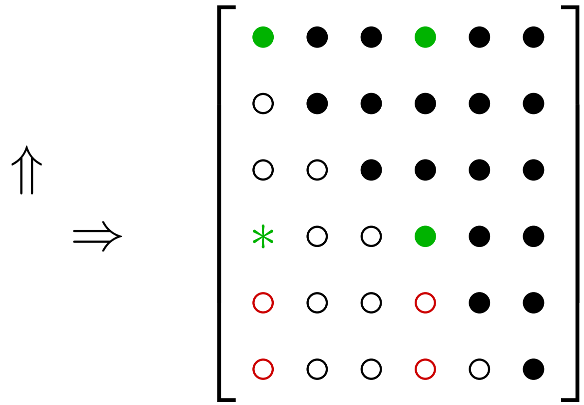
nonsymmetric case:
quadratic asymptotic convergence for “bottom-to-top”-sweeps,
but only linear asymptotic convergence for “top-to-bottom”-sweeps;

Surprise: sweep $\neq$ sweep



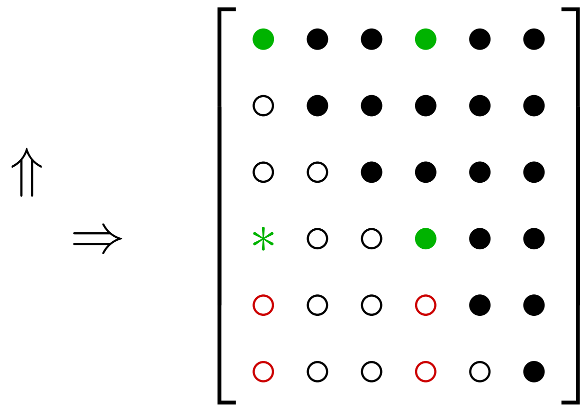
Heuristic explanation: in "top-to-bottom"-sweeps **elements** that have already been annihilated in the current sweep are recombined with potentially large elements from the upper triangular part.

Surprise: sweep $\neq$ sweep



Heuristic explanation: in "bottom-to-top"-sweeps such **elements** are recombined with usually small elements from the lower triangular part.

Surprise: sweep $\neq$ sweep



Theorem (M., 2008): The nonsymmetric Jacobi algorithm is asympotically quadratically convergent **if north-east-directed sweeps are used**.

north-east-directed sweep: the sequence of indices $((i_1, j_1), ..., (i_N, j_N))$, $N = n(n-1)/2$ in which order the elements are annihilated satisfies

$$\nu < \mu \quad \Rightarrow \quad (i_\nu > i_\mu \text{ or } j_\nu < j_\mu)$$

Nonsymmetric Jacobi method

Properties:

- „off-Norm“ $\text{off}(M) = \sqrt{\sum_{j<i} m_{ij}^2}$ is *not* reduces in each step
- experimentally: global convergence;
  open problem: proof of convergence
- experimentally: asymptotic linear or quadratic convergence;
  proof of local convergence for north-east-directed sweeps;
- fast if matrix close to Schur form
- in general much slower than QR (not competitive)

Method 2: A Jacobi-like method

Idea: Annihilate one diagonal or two off diagonal pivot elements in the strict upper anti-triangular part of Z in each Jacobi-step:

$$
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\
\cdot & {\color{red}\bullet} & \cdot & \cdot & \cdot & \cdot & * & * \\
\cdot & \cdot & \cdot & \cdot & \cdot & * & * & * \\
\cdot & \cdot & \cdot & \cdot & * & * & * & * \\
\cdot & \cdot & \cdot & * & * & * & * & * \\
\cdot & \cdot & * & * & * & * & * & * \\
\cdot & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & *
\end{bmatrix}
\qquad
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\
\cdot & \cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & * & * \\
\cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & * & * & * \\
\cdot & \cdot & \cdot & \cdot & * & * & * & * \\
\cdot & \cdot & \cdot & * & * & * & * & * \\
\cdot & \cdot & * & * & * & * & * & * \\
\cdot & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & *
\end{bmatrix}
$$

This can always be achieved via a unitary consimilarity transformation.

Method 2: A Jacobi-like method

Diagonal pivots:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & {\color{red}\bullet} & \cdot & \cdot & \cdot & \cdot & {\color{blue}\bullet} & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & \cdot & \cdot & * & * & * & * \\ \cdot & \cdot & \cdot & * & * & * & * & * \\ \cdot & \cdot & * & * & * & * & * & * \\ \cdot & {\color{blue}\bullet} & * & * & * & * & {\color{blue}\bullet} & * \\ * & * & * & * & * & * & * & * \end{bmatrix}$$

Consider the colored 2×2 subproblem:

$$\begin{bmatrix} {\color{red}\bullet} & {\color{blue}\bullet} \\ {\color{blue}\bullet} & {\color{blue}\bullet} \end{bmatrix}$$

Method 2: A Jacobi-like method

Diagonal pivots:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & {\color{red}\bullet} & \cdot & \cdot & \cdot & \cdot & {\color{blue}\bullet} & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & \cdot & \cdot & * & * & * & * \\ \cdot & \cdot & \cdot & * & * & * & * & * \\ \cdot & \cdot & * & * & * & * & * & * \\ \cdot & {\color{blue}\bullet} & * & * & * & * & {\color{blue}\bullet} & * \\ * & * & * & * & * & * & * & * \end{bmatrix}$$

Compute the anti-triangular form of the 2×2 problem:

$$\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix} \begin{bmatrix} {\color{red}\bullet} & {\color{blue}\bullet} \\ {\color{blue}\bullet} & {\color{blue}\bullet} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} {\color{red}\circ} & * \\ * & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Diagonal pivots:

$$
\begin{bmatrix}
1 & & & & & & & \\
 & u_{11} & & & & & u_{21} & \\
 & & 1 & & & & & \\
 & & & 1 & & & & \\
 & & & & 1 & & & \\
 & & & & & 1 & & \\
 & u_{12} & & & & & u_{22} & \\
 & & & & & & & 1
\end{bmatrix}
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\
\cdot & \color{red}{\bullet} & \cdot & \cdot & \cdot & \cdot & \color{blue}{\bullet} & * \\
\cdot & \cdot & \cdot & \cdot & \cdot & * & * & * \\
\cdot & \cdot & \cdot & \cdot & * & * & * & * \\
\cdot & \cdot & \cdot & * & * & * & * & * \\
\cdot & \cdot & * & * & * & * & * & * \\
\cdot & \color{blue}{\bullet} & * & * & * & * & \color{blue}{\bullet} & * \\
* & * & * & * & * & * & * & *
\end{bmatrix}
\begin{bmatrix}
1 & & & & & & & \\
 & u_{11} & & & & & u_{12} & \\
 & & 1 & & & & & \\
 & & & 1 & & & & \\
 & & & & 1 & & & \\
 & & & & & 1 & & \\
 & u_{21} & & & & & u_{22} & \\
 & & & & & & & 1
\end{bmatrix}
$$

Then update the $n \times n$ matrix.

$$
\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix}
\begin{bmatrix} \color{red}{\bullet} & \color{blue}{\bullet} \\ \color{blue}{\bullet} & \color{blue}{\bullet} \end{bmatrix}
\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}
=
\begin{bmatrix} \color{red}{\circ} & * \\ * & * \end{bmatrix}
$$

Method 2: A Jacobi-like method

Diagonal pivots:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \circ & \cdot & \cdot & \cdot & \cdot & * & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & \cdot & \cdot & * & * & * & * \\ \cdot & \cdot & \cdot & * & * & * & * & * \\ \cdot & \cdot & * & * & * & * & * & * \\ \cdot & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix}$$

Then update the $n \times n$ matrix.

$$\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix} \begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} \circ & * \\ * & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Off-diagonal pivots:

$$
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\
\cdot & \cdot & \bullet & \cdot & \cdot & \cdot & * & * \\
\cdot & \bullet & \cdot & \cdot & \cdot & * & * & * \\
\cdot & \cdot & \cdot & \cdot & * & * & * & * \\
\cdot & \cdot & \cdot & * & * & * & * & * \\
\cdot & \cdot & * & * & * & * & * & * \\
\cdot & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & *
\end{bmatrix}
$$

Question: Why consider two pivots?

Method 2: A Jacobi-like method

Off-diagonal pivots: assume, we only consider one off-diagonal pivot;

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & {\color{blue}\bullet} & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & \cdot & \cdot & * & * & * & * \\ \cdot & \cdot & \cdot & * & * & * & * & * \\ \cdot & \cdot & {\color{blue}\bullet} & * & * & * & {\color{blue}\bullet} & * \\ \cdot & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix}$$

Consider the colored 2×2 problem:

$$\begin{bmatrix} {\color{green}\bullet} & {\color{blue}\bullet} \\ {\color{blue}\bullet} & {\color{blue}\bullet} \end{bmatrix}$$

Method 2: A Jacobi-like method

Off-diagonal pivots: assume, we only consider one off-diagonal pivot;

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \color{green}{\bullet} & \cdot & \cdot & \cdot & \color{blue}{\bullet} & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & \cdot & \cdot & * & * & * & * \\ \cdot & \cdot & \cdot & * & * & * & * & * \\ \cdot & \cdot & \color{blue}{\bullet} & * & * & * & \color{blue}{\bullet} & * \\ \cdot & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix}$$

Compute the anti-triangular form of the 2×2 problem:

$$\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix} \begin{bmatrix} \color{green}{\bullet} & \color{blue}{\bullet} \\ \color{blue}{\bullet} & \color{blue}{\bullet} \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} \color{green}{\circ} & * \\ * & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Off-diagonal pivots: assume, we only consider one off-diagonal pivot;

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ast \\ \cdot & \cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & {\color{blue}\bullet} & \ast \\ \cdot & \cdot & \cdot & \cdot & \cdot & \ast & \ast & \ast \\ \cdot & \cdot & \cdot & \cdot & \ast & \ast & \ast & \ast \\ \cdot & \cdot & \cdot & \ast & \ast & \ast & \ast & \ast \\ \cdot & \cdot & {\color{blue}\bullet} & \ast & \ast & \ast & {\color{blue}\bullet} & \ast \\ \cdot & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \end{bmatrix}$$

We may use different unitary transformation from the left and the right,

$$\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix} \begin{bmatrix} {\color{green}\bullet} & {\color{blue}\bullet} \\ {\color{blue}\bullet} & {\color{blue}\bullet} \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} {\color{green}\circ} & \ast \\ \ast & \ast \end{bmatrix}$$

Method 2: A Jacobi-like method

Off-diagonal pivots: assume, we only consider one off-diagonal pivot;

$$
\begin{bmatrix}
1 & & & & & & & \\
 & u_{11} & & & & u_{21} & & \\
 & & v_{11} & & & & v_{21} & \\
 & & & 1 & & & & \\
 & & & & 1 & & & \\
 & u_{12} & & & & u_{22} & & \\
 & & v_{12} & & & & v_{22} & \\
 & & & & & & & 1
\end{bmatrix}
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ast \\
\cdot & \cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & {\color{blue}\bullet} & \ast \\
\cdot & \cdot & \cdot & \cdot & \cdot & \ast & \ast & \ast \\
\cdot & \cdot & \cdot & \cdot & \ast & \ast & \ast & \ast \\
\cdot & \cdot & \cdot & \ast & \ast & \ast & \ast & \ast \\
\cdot & \cdot & {\color{blue}\bullet} & \ast & \ast & \ast & {\color{blue}\bullet} & \ast \\
\cdot & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast
\end{bmatrix}
\begin{bmatrix}
1 & & & & & & & \\
 & u_{11} & & & & u_{12} & & \\
 & & v_{11} & & & & v_{12} & \\
 & & & 1 & & & & \\
 & & & & 1 & & & \\
 & u_{21} & & & & u_{22} & & \\
 & & v_{21} & & & & v_{22} & \\
 & & & & & & & 1
\end{bmatrix}
$$

because of the position of the subproblem. $\leadsto$ more freedom in the parameters

$$
\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix}
\begin{bmatrix} {\color{green}\bullet} & {\color{blue}\bullet} \\ {\color{blue}\bullet} & {\color{blue}\bullet} \end{bmatrix}
\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}
=
\begin{bmatrix} {\color{green}\circ} & \ast \\ \ast & \ast \end{bmatrix}
$$

Method 2: A Jacobi-like method

Off-diagonal pivots: assume, we only consider one off-diagonal pivot;

$$
\begin{bmatrix}
1 & & & & & & & \\
 & u_{11} & & & & u_{21} & & \\
 & & v_{11} & & & & v_{21} & \\
 & & & 1 & & & & \\
 & & & & 1 & & & \\
 & u_{12} & & & & u_{22} & & \\
 & & v_{12} & & & & v_{22} & \\
 & & & & & & & 1
\end{bmatrix}
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\
\cdot & \cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & {\color{blue}\bullet} & * \\
\cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & \bullet & * & * \\
\cdot & \cdot & \cdot & \cdot & * & * & * & * \\
\cdot & \cdot & \cdot & * & * & * & * & * \\
\cdot & \cdot & {\color{blue}\bullet} & * & * & * & {\color{blue}\bullet} & * \\
\cdot & \bullet & * & * & * & \bullet & * & * \\
* & * & * & * & * & * & * & *
\end{bmatrix}
\begin{bmatrix}
1 & & & & & & & \\
 & u_{11} & & & & u_{12} & & \\
 & & v_{11} & & & & v_{12} & \\
 & & & 1 & & & & \\
 & & & & 1 & & & \\
 & u_{21} & & & & u_{22} & & \\
 & & v_{21} & & & & v_{22} & \\
 & & & & & & & 1
\end{bmatrix}
$$

Simultaneously, a second 2×2 system – marked by $\bullet$ – will be transformed.

$$
\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix}
\begin{bmatrix} {\color{green}\bullet} & {\color{blue}\bullet} \\ {\color{blue}\bullet} & {\color{blue}\bullet} \end{bmatrix}
\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}
=
\begin{bmatrix} {\color{green}\circ} & * \\ * & * \end{bmatrix}
$$

$$
\begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix}
\begin{bmatrix} {\color{green}\bullet} & \bullet \\ \bullet & \bullet \end{bmatrix}
\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}
=
\begin{bmatrix} {\color{green}*} & * \\ * & * \end{bmatrix}
$$

Method 2: A Jacobi-like method

Off-diagonal pivots: assume, we only consider one off-diagonal pivot;

$$
\begin{bmatrix}
1 & & & & & & & \\
 & u_{11} & & & & u_{21} & & \\
 & & v_{11} & & & & v_{21} & \\
 & & & 1 & & & & \\
 & & & & 1 & & & \\
 & u_{12} & & & & u_{22} & & \\
 & & v_{12} & & & & v_{22} & \\
 & & & & & & & 1
\end{bmatrix}
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ast \\
\cdot & \cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & {\color{blue}\bullet} & \ast \\
\cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & \bullet & \ast & \ast \\
\cdot & \cdot & \cdot & \cdot & \ast & \ast & \ast & \ast \\
\cdot & \cdot & \cdot & \ast & \ast & \ast & \ast & \ast \\
\cdot & \cdot & {\color{blue}\bullet} & \ast & \ast & \ast & {\color{blue}\bullet} & \ast \\
\cdot & \bullet & \ast & \ast & \ast & \bullet & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast
\end{bmatrix}
\begin{bmatrix}
1 & & & & & & & \\
 & u_{11} & & & & u_{12} & & \\
 & & v_{11} & & & & v_{12} & \\
 & & & 1 & & & & \\
 & & & & 1 & & & \\
 & u_{21} & & & & u_{22} & & \\
 & & v_{21} & & & & v_{22} & \\
 & & & & & & & 1
\end{bmatrix}
$$

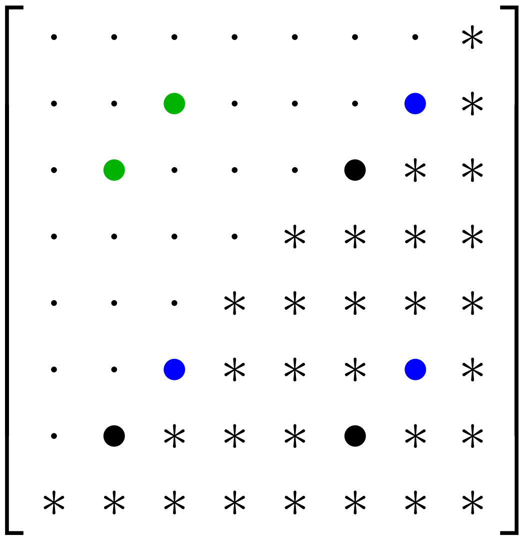
Use the freedom in the parameters to anti-triangularize the second system as well.

$$
\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix}
\begin{bmatrix} {\color{green}\bullet} & {\color{blue}\bullet} \\ {\color{blue}\bullet} & {\color{blue}\bullet} \end{bmatrix}
\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}
=
\begin{bmatrix} {\color{green}\circ} & \ast \\ \ast & \ast \end{bmatrix}
$$

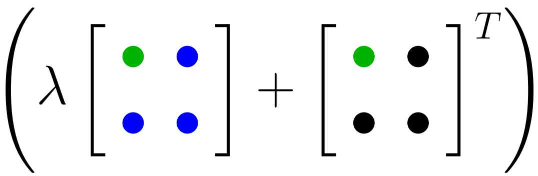
$$
\begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix}
\begin{bmatrix} {\color{green}\bullet} & \bullet \\ \bullet & \bullet \end{bmatrix}
\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}
=
\begin{bmatrix} {\color{green}\ast} & \ast \\ \ast & \ast \end{bmatrix}
$$

Method 2: A Jacobi-like method

Off-diagonal pivots:

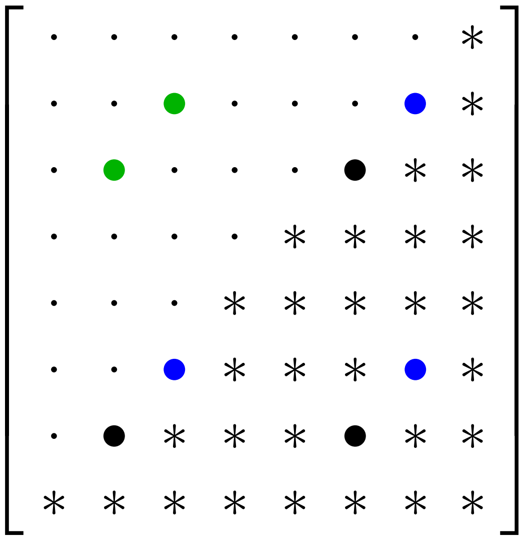


Anti-triangularize the colored/black generalized 2×2 problem:



Method 2: A Jacobi-like method

Off-diagonal pivots:



Anti-triangularize the colored/black generalized 2×2 problem:

$$\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix} \left(\lambda \begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \end{bmatrix} + \begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \end{bmatrix}^T \right) \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \lambda \begin{bmatrix} \circ & * \\ * & * \end{bmatrix} + \begin{bmatrix} \circ & * \\ * & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Off-diagonal pivots:

$$
\begin{bmatrix}
1 &&&&&&& \\
& u_{11} &&&& u_{21} && \\
&& v_{11} &&&& v_{21} & \\
&&& 1 &&&& \\
&&&& 1 &&& \\
& u_{12} &&&& u_{22} && \\
&& v_{12} &&&& v_{22} & \\
&&&&&&& 1
\end{bmatrix}
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\
\cdot & \cdot & \color{green}{\bullet} & \cdot & \cdot & \cdot & \color{blue}{\bullet} & * \\
\cdot & \color{green}{\bullet} & \cdot & \cdot & \cdot & \bullet & * & * \\
\cdot & \cdot & \cdot & \cdot & * & * & * & * \\
\cdot & \cdot & \cdot & * & * & * & * & * \\
\cdot & \cdot & \color{blue}{\bullet} & * & * & * & \color{blue}{\bullet} & * \\
\cdot & \bullet & * & * & * & \bullet & * & * \\
* & * & * & * & * & * & * & *
\end{bmatrix}
\begin{bmatrix}
1 &&&&&&& \\
& u_{11} &&&& u_{12} && \\
&& v_{11} &&&& v_{12} & \\
&&& 1 &&&& \\
&&&& 1 &&& \\
& u_{21} &&&& u_{22} && \\
&& v_{21} &&&& v_{22} & \\
&&&&&&& 1
\end{bmatrix}
$$

Update the $n \times n$ matrix.

$$
\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix}
\left(\lambda \begin{bmatrix} \color{green}{\bullet} & \color{blue}{\bullet} \\ \color{blue}{\bullet} & \color{blue}{\bullet} \end{bmatrix} + \begin{bmatrix} \color{green}{\bullet} & \bullet \\ \bullet & \bullet \end{bmatrix}^T \right)
\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}
= \lambda \begin{bmatrix} \color{green}{\circ} & * \\ * & * \end{bmatrix} + \begin{bmatrix} \color{green}{\circ} & * \\ * & * \end{bmatrix}
$$

Method 2: A Jacobi-like method

Off-diagonal pivots:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ast \\ \cdot & \cdot & \circ & \cdot & \cdot & \cdot & \ast & \ast \\ \cdot & \circ & \cdot & \cdot & \cdot & \ast & \ast & \ast \\ \cdot & \cdot & \cdot & \cdot & \ast & \ast & \ast & \ast \\ \cdot & \cdot & \cdot & \ast & \ast & \ast & \ast & \ast \\ \cdot & \cdot & \ast & \ast & \ast & \ast & \ast & \ast \\ \cdot & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \end{bmatrix}$$

Update the $n \times n$ matrix.

$$\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix} \left(\lambda \begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \end{bmatrix} + \begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \end{bmatrix}^T \right) \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \lambda \begin{bmatrix} \circ & \ast \\ \ast & \ast \end{bmatrix} + \begin{bmatrix} \circ & \ast \\ \ast & \ast \end{bmatrix}$$

Method 2: A Jacobi-like method

Sweep: Annihilate each pivot element at least once.

E.g., cyclic-by-row-sweep:

$$\begin{bmatrix} \circ & \cdot & \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & * & * \\ \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & * & * & * & * \\ \cdot & * & * & * & * & * \\ \bullet & * & * & * & * & \bullet \end{bmatrix}$$

Method 2: A Jacobi-like method

Sweep: Annihilate each pivot element at least once.

E.g., cyclic-by-row-sweep:

$$\begin{bmatrix} \cdot & \circ & \cdot & \cdot & \cdot & \bullet \\ \circ & \cdot & \cdot & \cdot & \bullet & * \\ \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & * & * & * & * \\ \cdot & \bullet & * & * & * & \bullet \\ \bullet & * & * & * & \bullet & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Sweep: Annihilate each pivot element at least once.

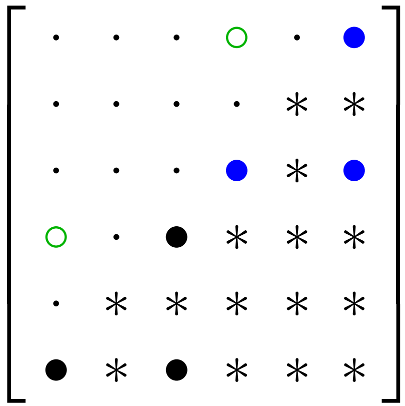
E.g., cyclic-by-row-sweep:

$$\begin{bmatrix} \cdot & \cdot & \circ & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & * & * \\ \circ & \cdot & \cdot & \bullet & * & * \\ \cdot & \cdot & \bullet & * & * & \bullet \\ \cdot & * & * & * & * & * \\ \bullet & * & * & \bullet & * & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Sweep: Annihilate each pivot element at least once.

E.g., cyclic-by-row-sweep:



Method 2: A Jacobi-like method

Sweep: Annihilate each pivot element at least once.

E.g., cyclic-by-row-sweep:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \circ & \bullet \\ \cdot & \cdot & \cdot & \cdot & \bullet & \bullet \\ \cdot & \cdot & \cdot & \ast & \ast & \ast \\ \cdot & \cdot & \ast & \ast & \ast & \ast \\ \circ & \bullet & \ast & \ast & \ast & \ast \\ \bullet & \bullet & \ast & \ast & \ast & \ast \end{bmatrix}$$

Method 2: A Jacobi-like method

Sweep: Annihilate each pivot element at least once.

E.g., cyclic-by-row-sweep:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \circ & \cdot & \cdot & \bullet & * \\ \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & * & * & * & * \\ \cdot & \bullet & * & * & \bullet & * \\ * & * & * & * & * & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Sweep: Annihilate each pivot element at least once.

E.g., cyclic-by-row-sweep:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \circ & \cdot & \bullet & * \\ \cdot & \circ & \cdot & \bullet & * & * \\ \cdot & \cdot & \bullet & * & \bullet & * \\ \cdot & \bullet & * & \bullet & * & * \\ * & * & * & * & * & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Sweep: Annihilate each pivot element at least once.

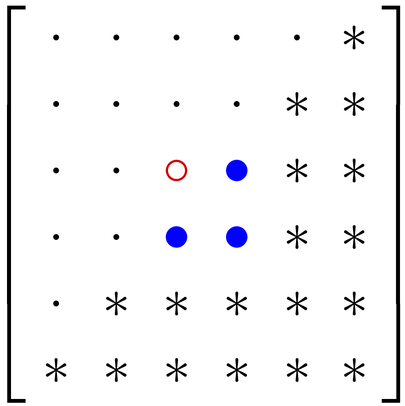
E.g., cyclic-by-row-sweep:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & \circ & \bullet & * \\ \cdot & \cdot & \cdot & \bullet & \bullet & * \\ \cdot & \circ & \bullet & * & * & * \\ \cdot & \bullet & \bullet & * & * & * \\ * & * & * & * & * & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Sweep: Annihilate each pivot element at least once.

E.g., cyclic-by-row-sweep:



Method 2: A Jacobi-like method

Properties of the algorithm:

+ locally and asymptotically quadratically convergent;

+ globally convergent in experiments;

+ converges fast for matrices Z close to anti-triangular form

− expensive in general (cost of 3 sweeps $\hat{=}$ cost of `QZ`)

− convergence problems for badly scaled problems

− convergence problems for large n

Method No. 3: A Hybrid Method

Method 3: A hybrid method

Laub-trick:

$+$ works for moderate sizes of n;

$+$ essentially cost of `QZ`;

$-$ problems for eigenvalues with modulus near one;

Jacobi:

$+$ works nicely if problem is small and eigenvalues do not differ too much in modulus;

Idea: Combine the positive properties of these two algorithms. Use the Laub-trick for getting all eigenvalues sufficiently far away from the unit circle and use Jacobi for the eigenvalues near the unit circle.

Method 3: A hybrid method

$$\lambda Z + Z^T = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \left(\lambda \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix} + \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ 0 & Y_{22} & Y_{23} \\ 0 & 0 & Y_{33} \end{bmatrix} \right) \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix}$$

Step 1: Given a tolerance $\alpha > 1$ and a regular $\lambda Z + Z^T \in \mathbb{C}^{2n \times 2n}$, compute its generalized Schur decomposition, where the eigenvalues are ordered in such a way that

$$\begin{aligned} \sigma(\lambda X_{11} + Y_{11}) &\subseteq \{\lambda \in \mathbb{C} : |\lambda| \geq \alpha\}, \\ \sigma(\lambda X_{22} + Y_{22}) &\subseteq \{\lambda \in \mathbb{C} : \alpha > |\lambda| > \frac{1}{\alpha}\}, \\ \sigma(\lambda X_{33} + Y_{33}) &\subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{\alpha}\}. \end{aligned}$$

Method 3: A hybrid method

$$\lambda Z + Z^T = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \left(\lambda \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix} + \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ 0 & Y_{22} & Y_{23} \\ 0 & 0 & Y_{33} \end{bmatrix} \right) \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix}$$

Step 2: By the Laub trick, the matrix

$$\begin{bmatrix} W_{11} & Q_{11}^T R_m \\ W_{21} & Q_{12}^T R_m \\ W_{31} & Q_{13}^T R_m \end{bmatrix}$$

has orthonormal columns. Extend this matrix to a unitary matrix

$$U := \begin{bmatrix} W_{11} & U_{12} & Q_{11}^T R_m \\ W_{21} & U_{22} & Q_{12}^T R_m \\ W_{31} & U_{32} & Q_{13}^T R_m \end{bmatrix}.$$

Method 3: A hybrid method

$$\lambda Z + Z^T = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \left(\lambda \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix} + \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ 0 & Y_{22} & Y_{23} \\ 0 & 0 & Y_{33} \end{bmatrix} \right) \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix}$$

Step 3: Compute

$$U^T Z U = \begin{bmatrix} 0 & 0 & Y_{11}^T R_m \\ 0 & Z_{22} & Z_{23} \\ R_m X_{11} & Z_{32} & Z_{33} \end{bmatrix},$$

where $Y_{11}^T R_m \in \mathbb{C}^{m \times m}$ and $R_m X_{11} \in \mathbb{C}^{m \times m}$ are in anti-triangular form and $Z_{22} \in \mathbb{C}^{(n-2m)\times(n-2m)}$ has only eigenvalues in $\{\lambda \in \mathbb{C} : \alpha > |\lambda| > \frac{1}{\alpha}\}$.

Method 3: A hybrid method

$$\lambda Z + Z^T = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \left(\lambda \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix} + \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ 0 & Y_{22} & Y_{23} \\ 0 & 0 & Y_{33} \end{bmatrix} \right) \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix}$$

Step 3: Compute

$$U^T Z U = \begin{bmatrix} 0 & 0 & Y_{11}^T R_m \\ 0 & Z_{22} & Z_{23} \\ R_m X_{11} & Z_{32} & Z_{33} \end{bmatrix},$$

where $Y_{11}^T R_m \in \mathbb{C}^{m \times m}$ and $R_m X_{11} \in \mathbb{C}^{m \times m}$ are in anti-triangular form and $Z_{22} \in \mathbb{C}^{(n-2m)\times(n-2m)}$ has only eigenvalues in $\{\lambda \in \mathbb{C} : \alpha > |\lambda| > \frac{1}{\alpha}\}$.

Step 4: Anti-triangularize Z_{22} by some expensive, but accurate method (e.g., palindromic Jacobi algorithm, Schröder's palindromic QR algorithm).

Method No. 4: Palindromic QR

Method No. 4: Palindromic QR

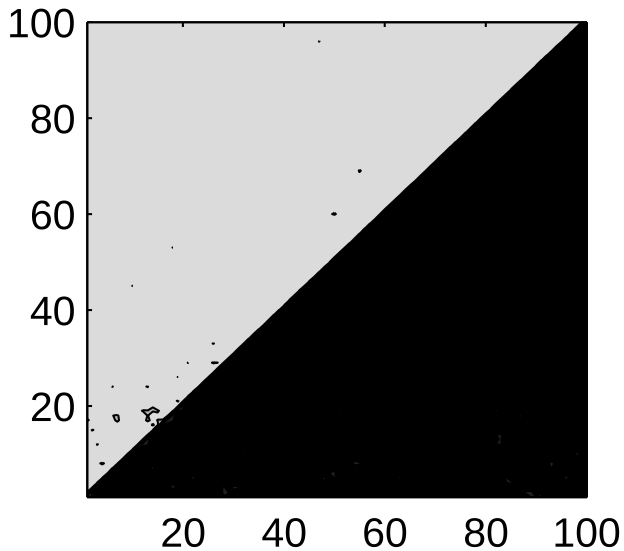
Ph.D. thesis by Christian Schröder (2008) – too much for this talk

Numerical experiments

Numerical experiments

Test: 100 random 100×100 matrices with 10 eigenvalues in an annulus in the complex plane with outer radius $1+10^{-12}$ and inner radius $1/(1+10^{-12})$.

Typical behavior: (black: elements of modulus larger than one; light grey: elements of modulus 10^{-15})

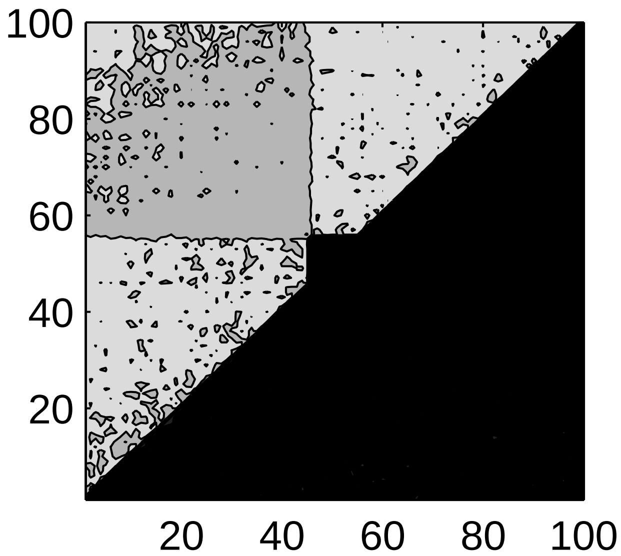


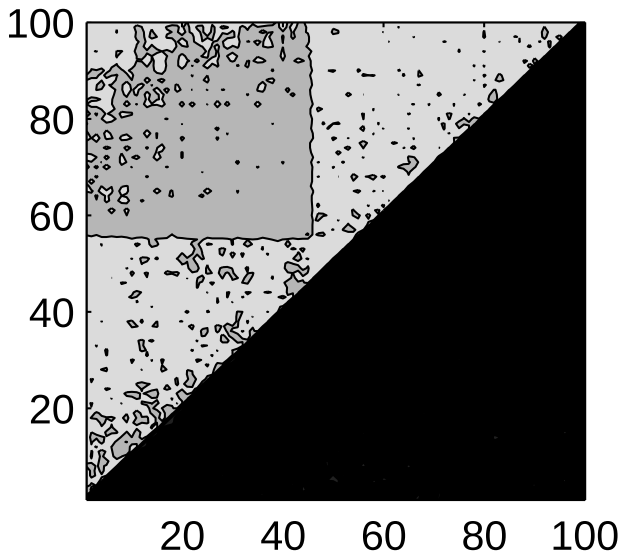
similar results when solving the small subproblem with palindromic Jacobi or the palindromic QR

Numerical experiments

Tough test: random 100×100 matrices with 10 eigenvalues in a disk with radius 10^{-12} and center 1.

Typical behavior: (black: elements of modulus larger than one; light grey: elements of modulus 10^{-15})



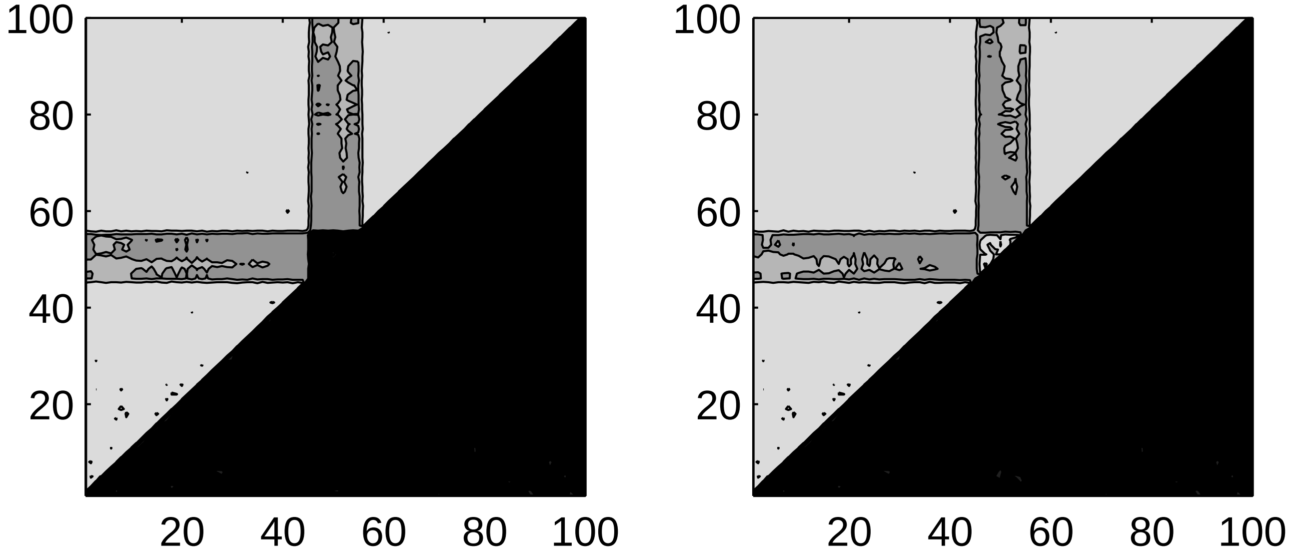


left: method 4 before applying palindromic QR to the small problem
right: method 4 before applying palindromic QR to the small problem

Numerical experiments

Tough test: random 100×100 matrices with 10 eigenvalues in a disk with radius 10^{-12} and center 1.

Typical behavior: (black: elements of modulus larger than one; light grey: elements of modulus 10^{-15})

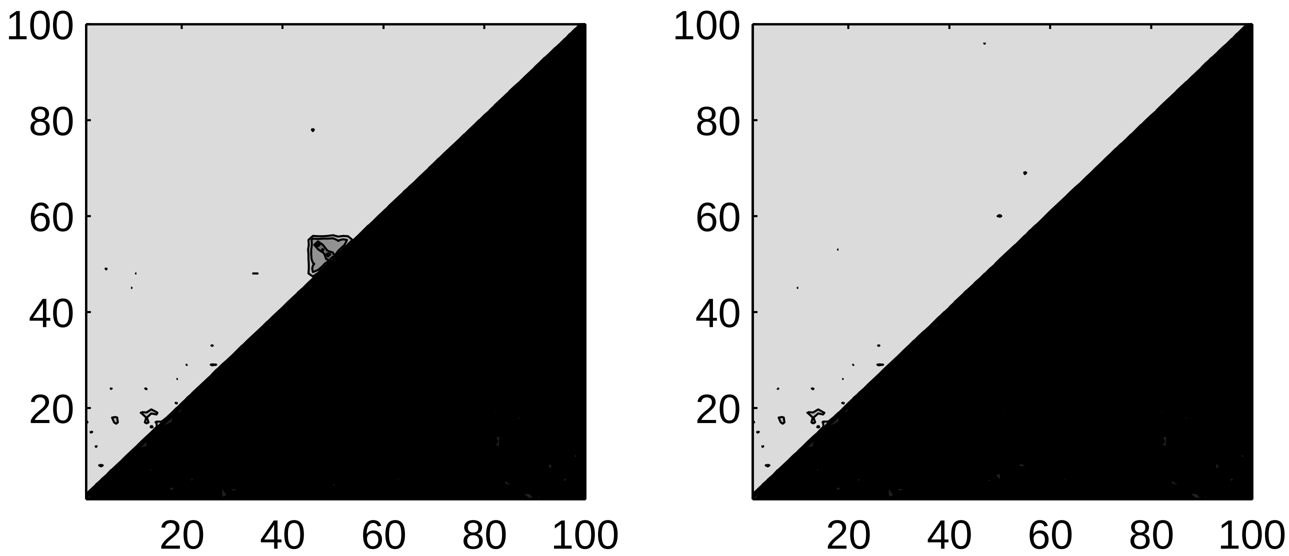


left: method 4 followed by a full sweep of Jacobi before...
right: ...and after solving the small problem by the palindromic QR

Numerical experiments

Tough test: random 100×100 matrices with 10 eigenvalues in a disk with radius 10^{-12} and center 1.

Typical behavior: (black: elements of modulus larger than one; light grey: elements of modulus 10^{-15})



left: method 4 with palindromic QR, then full Jacobi sweep before...
right: and after solving the small problem once again with palindromic QR

Thank you for your attention!