Inverse Problems for Selfadjoint Matrix Polynomials

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Preliminaries

Given $A_0, A_1, \cdots, A_l \in \mathbb{C}^{n\times n}$ (or possibly in $\mathbb{R}^{n\times n}$),

$$L(\lambda) := \sum_{j=0}^{l} A_j \lambda^j, \qquad \lambda \in \mathbb{C},$$

$$\det A_l \neq 0.$$

Spectrum of L : $\sigma(L) := \{\lambda \in \mathbb{C} : \det L(\lambda) = 0\}$ (the eigenvalues).

Eigenvectors: If $\lambda_0 \in \sigma(L)$ vectors $x \in \mathbb{C}^n$ such that $x \neq 0$ and $L(\lambda_0)x = 0$ are eigenvectors.

Direct problems: Given $L(\lambda)$ find $\sigma(L)$, eigenvectors, etc..

Inverse problems: Given a set of candidates for *eigenvalues* and *eigenvectors* (possibly incomplete) find system(s) $L(\lambda)$ consistent with this spectral data.

The ultimate inverse problem:

One can argue that the "ultimate" inverse problem is the expression of the coefficients A_j in terms of spectral data.

i.e. when the spectral data is properly (completely) defined it should determine the coefficients of the system.

Hence our interest in canonical forms.

Important special cases

1. $l = 1$.
2. $l = 2$: the quadratic eigenvalue problem.
3. $A_j \in \mathbb{R}^{n\times n}$ for all j.
4. $A_j^* = A_j \in \mathbb{C}^{n\times n}$ for all j, $A_l > 0$? (et al?)
An equivalent first-degree problem is selfadjoint in a natural inner-product on $\mathbb{C}^{ln\times ln}$. We say that $L(\lambda)$ is selfadjoint.
5. $A_j^T = A_j \in \mathbb{R}^{n\times n}$ for all j, $A_l > 0$? (et al?)
Selfadjoint...on $\mathbb{R}^{ln\times ln}$.

Note that the role of eigenvalues of $L(\lambda)$ as the singularitites of $L(\lambda)^{-1}$ is of great importance.

Sign characteristics (from GLR, 2005).

Theorem

Let $L(\lambda)$ be an $n \times n$ self-adjoint matrix polynomial with non-singular leading coefficient and let $\mu_1(\lambda), \ldots, \mu_n(\lambda)$ be the real analytic functions of real λ such that

$$\det\{\mu_j(\lambda)I_n - L(\lambda)\} = 0 \quad \text{for} \quad j = 1, \ldots, n.$$

Let $\lambda_1 < \cdots < \lambda_r$ be the different real eigenvalues of $L(\lambda)$ (zeros of det$L(\lambda)$)). For $j = 1, 2, \ldots, n$ and every $i = 1, \ldots, r$, write

$$\mu_j(\lambda) = (\lambda - \lambda_i)^{m_{ij}} \nu_{ij}(\lambda) \quad \text{where} \quad \nu_{ij}(\lambda_i) \neq 0 \quad \text{is real.}$$

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Then the non-zero numbers among $m_{i1}, \ldots, m_{in}$ *are the partial multiplicities of* $L(\lambda)$ *associated with* λ_i*, and the sign of* $\nu_{ij}(\lambda_i)$ *(for* $m_{ij} \neq 0$*) is the sign characteristic associated with the elementary divisors* $(\lambda - \lambda_i)^{m_{ij}}$ *of* $L(\lambda)$.

Sign characteristics and inverse problems

Spectral properties for **selfadjoint** matrix functions include **sign characteristics** - as above. So they have a role to play in the formulation of inverse problems.

For example, suppose that the leading coefficient A_l is NOT positive definite, and consider the behaviour of the eigenfunctions $\mu_j(\lambda)$ as $\lambda \to \infty$.

Theorem

Let $(\pi, n - \pi, 0)$ be the inertia *of L_l and assume that there exists at least one real eigenvalue of $L(\lambda)$. Let λ_{max} be the largest real eigenvalue of $L(\lambda)$.*

Then there are π indices $\{i_1, \ldots, i_\pi\}$ in $\{1, 2, \ldots, n\}$ such that, for all $\lambda > \lambda_{max}$,

$$\mu_j(\lambda) > 0 \quad \text{if} \quad j \in \{i_1, \ldots, i_\pi\} \quad \textit{and} \quad \mu_j(\lambda) < 0 \quad \text{if} \quad j \notin \{i_1, \ldots, i_\pi\}.$$

Quadratics using divisors (with Tisseur and then Chorianopoulos)

Consider quadratic functions with Hermitian coeffts.:

$$L(\lambda) = L_2\lambda^2 + L_1\lambda + L_0, \qquad \det L_2 \neq 0.$$

We would like to express $L(\lambda)$ in *factored* form,

$$L(\lambda) = (I\lambda - S)L_2(I\lambda - A).$$

More precisely, given nonsingular Hermitian L_2 and the *right divisor* $I\lambda - A$, describe the class of matrices S for which $L(\lambda)$ is selfadjoint.

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First define $T_H = \{A^*L_2 + Z_1 \;:\; A^*Z_1 - Z_1A = 0 \text{ and } Z_1^* = Z_1\}$.

Theorem

Given L_2 nonsingular and a right divisor $I\lambda - A$, the function $L(\lambda)$ is Hermitian when $\lambda \in \mathbb{R}$ if and only if $S = TL_2^{-1}$ and $T \in T_H$.

Example

Nonsingular leading coefficient:

$$L_2 = \begin{bmatrix} 2 & 5 & 6 & 2 & 0 \\ * & 15 & 19 & 9 & 12 \\ * & * & 23 & 10 & 18 \\ * & * & * & 5 & 6 \\ * & * & * & * & 35 \end{bmatrix}.$$

(Diagonal) right divisor $I\lambda - A$ where

$$A = \mathrm{diag}\left[\ 1,\quad 3,\quad 4,\quad 2+i,\quad 5-3i\ \right].$$

Calculated left divisor S has spectrum

$$1.5454, \quad 3.4996 \pm 0.4481i, \quad 2-i, \quad 5+3i.$$

Updating problems: Given $L(\lambda)$ and a *complete* description of $\sigma(L)$, adjust the coefficients of $L(\lambda)$ to produce desired changes in $\sigma(L)$.

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Standard pair: $X \in \mathcal{F}^{n\times ln}$, $T \in \mathcal{F}^{ln\times ln}$, ($\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$) for which

$$\det \begin{bmatrix} X \\ XT \\ \vdots \\ XT^{l-1} \end{bmatrix} \neq 0.$$

Standard pair for $L(\lambda)$ is a standard pair (X, T) for which

$$L(X,T) := L_l XT^l + \cdots + L_1 XT + L_0 X = 0.$$

With T in Jordan form, columns of X determine eigenvectors (and generalized eigenvectors) of $L(\lambda)$.

From pairs to triples

If the leading coefft. L_l is invertible, then a triple (X, T, Y) is a standard triple for $L(\lambda)$ if (X, T) is a standard pair for $L(\lambda)$ and

$$Y = \begin{bmatrix} X \\ XT \\ \vdots \\ XT^{l-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ {L_l}^{-1} \end{bmatrix}.$$

Similarity of standard triples:

$$\{\text{Standard Triples}\} = \{XS,\ S^{-1}TS,\ S^{-1}Y\} \ : \ (X, T, Y) \text{ is a st. triple.}\}$$

JORDAN forms → JORDAN pairs → JORDAN triples.

Hermitian/real-symmetric systems

Symmetries in coefficients $\leftrightarrow$ Symmetries in canonical structures

Prime example of a standard triple:

$$X = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}, \quad T = C_R, \quad Y = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ L_l^{-1} \end{bmatrix},$$

where C_R is the (right) companion matrix of $L(\lambda)$.
Any triple *similar* to the above is also a standard triple..

Symmetries in terms of standard triples

Definition:
(a) A real standard triple (X, T, Y) is real self-adjoint if there is a real nonsingular H such that

$$Y^T = XH^{-1}, \quad T^T = HTH^{-1}, \quad X^T = HY.$$

(b) A complex standard triple (X, T, Y) is self-adjoint if there is a nonsingular Hermitian H such that

$$Y^* = XH^{-1}, \quad T^* = HTH^{-1}, \quad X^* = HY.$$

Theorem

Let $L(\lambda)$ have real coefficients with A_l nonsingular. Then:
(a) If $L(\lambda)$ admits a real selfadjoint standard triple, then $L(\lambda)$ is real and symmetric.
(b) If $L(\lambda)$ is real and symmetric then all its real standard triples are selfadjoint.

APPLICATION - AN UPDATING PROBLEM

$M, D, K \in \mathbb{R}^{n\times n}$ all real-symmetric, $M > 0$.

$$L(\lambda) = M\lambda^2 + D\lambda + K.$$

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All e.v. distinct. Jordan pair:

$$J = \begin{bmatrix} U_1 + iW & 0 & 0 & 0 \\ 0 & U_2 & 0 & 0 \\ 0 & 0 & U_3 & 0 \\ 0 & 0 & 0 & U_1 - iW \end{bmatrix},$$

$$X = \begin{bmatrix} X_c & X_{R1} & X_{R2} & \bar{X}_c \end{bmatrix}.$$

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To complete the selfadjoint Jordan triple (X, J, PX^*) we need:

$$P = \begin{bmatrix} 0 & 0 & 0 & I_{n-r} \\ 0 & I_r & 0 & 0 \\ 0 & 0 & -I_r & 0 \\ I_{n-r} & 0 & 0 & 0 \end{bmatrix}.$$

Updating - contd..

Define the moments of the system:

$$\Gamma_j = X(J^j P)X^* \in \mathbb{C}^{n\times n} \quad j = 1, 2, 3, \ldots,$$

then

$$M = \Gamma_1^{-1}, \quad D = -M\Gamma_2 M, \quad K = -M\Gamma_3 M + D\Gamma_1 D.$$

Updating - contd..

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This gives a formal recursive solution to the inverse problem:
Given the spectral data in the form of the Jordan triple (X, J, PX^*), these formulae generate the coefficients of the system.

Updating - STRATEGY:

- Given a system with real symmetric $M,\ D,\ K$ and $M > 0$, compute a Jordan triple (X, J, PX^*).
- Make the updates in X and J to produce $\hat{X},\ \hat{J}$ so that:
  (a) the canonical matrix P is not disturbed, and
  (b) conditions $XPX^* = 0$ and $X(JP)X^* > 0$ are maintained.
- Compute the moments defined by $(\hat{X}, P, \hat{J})$ and hence new coefficients $\hat{M}, \hat{D}, \hat{K}$.

A CANONICAL selfadjoint triple for REAL, SEMISIMPLE systems with A_l **nonsingular**.

1. Should display ALL eigenvalue/sign-characteristic/eigenvector information in convenient form.

2. Found by searching among the similarity-class of all selfadjoint triples.

SPECTRAL DATA:

Let δ be the signature of leading coefft. L_l,
and $\chi = 0$ or 1 according as l is even or odd.

Let $r_1, \ldots, r_{q+\chi\delta}$ be the real eigenvalues of positive type, $r_{q+\chi\delta+1}, \ldots, r_{2q+\chi\delta}$ be the real e.v. of negative type and construct diagonal matrices of size $q + \chi\delta$ and q:

$$R_+ = diag[r_1, \ldots, r_{q+\chi\delta}],$$

$$R_- = diag[r_{q+\chi\delta+1}, \ldots, r_{2(q+\chi\delta)}].$$

A CANONICAL selfadjoint triple for REAL, SEMISIMPLE systems with A_l **nonsingular**

Write the $2s$ **conjugate pairs** of **eigenvalues** as follows:

$$\beta_j = \mu_j + i\nu_j, \qquad \beta_{j+1} = \bar{\beta}_j = \mu_j - i\nu_j \qquad (\nu_j > 0), \qquad j = 1, 3, \ldots, 2s-1,$$

and set

$$M = diag[\mu_1, \mu_3, \ldots, \mu_{2s-1}], \quad N = diag[\nu_1, \nu_3, \ldots, \nu_{2s-1}].$$

A CANONICAL TRIPLE (L., Prells and Zaballa, 2012)

If non-real eigenvectors are $u_j \pm iv_j$, for $j = 1, 2, \cdots, s$ they can be normalized in such a way that, with

$$V = [v_1 \cdots v_s],\ U = [u_1 \cdots u_s] \quad \in \quad \mathbb{R}^{n \times s}$$

there is a REAL (CANONICAL) STANDARD TRIPLE: (X, J, PX^T) with

$$J = \Pi^T J_R \Pi = \mathrm{Diag}(R_+, R_-, \begin{bmatrix} M & -N \\ N & M \end{bmatrix}) \in \mathbb{R}^{n\ell \times n\ell},$$

$$P = \Pi^T P_R \Pi = \mathrm{Diag}\,(I_{q+\chi\delta}, -I_q, -I_s, I_s) \in \mathbb{R}^{n\ell \times n\ell},$$

$$X = X_R \Pi = \begin{bmatrix} X_+ & X_- & V & U \end{bmatrix} \in \mathbb{R}^{n \times n\ell}.$$

(Still in the semisimple case. Admits real and complex spectrum.)

FIRST PROPERTIES

Moment conditions:

$$XJ^kPX^T = 0, \qquad k = 0, 1, \ldots, l-2,$$

$$XJ^{(l-1)}PX^T = L_l^{-1}.$$

Resolvent form:

$$\lambda^r L(\lambda)^{-1} = \begin{cases} XJ^r(I\lambda - J)^{-1}PX^T, & r = 0, 1, \ldots, \ell - 1, \\ XJ^\ell(I\lambda I - J)^{-1}PX^T + L_\ell^{-1}, & r = \ell, \end{cases}$$

CASE OF EVEN DEGREE: $l = 2m$:

Moment conditions can be written:

$$\begin{bmatrix} X \\ XJ \\ \vdots \\ XJ^{m-1} \end{bmatrix} P \begin{bmatrix} X^T & J^T X^T & \cdots & (J^T)^{m-1} X^T \end{bmatrix} = 0.$$

Now separate real spectrum of +ve and -ve types - along with separation of conjugate pairs (ref canonical forms) and define:

$$A = \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_{m-1} \end{bmatrix} = \begin{bmatrix} X_+ & U_0 \\ X_+ R_+ & U_1 \\ \vdots & \vdots \\ X_+ R_+^{m-1} & U_{m-1} \end{bmatrix}, \quad B = \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_{m-1} \end{bmatrix} = \begin{bmatrix} X_- & V_0 \\ X_- R_- & V_1 \\ \vdots & \vdots \\ X_- R_-^{m-1} & V_{m-1} \end{bmatrix}.$$

CASE OF EVEN DEGREE (the punch-line):

With A and B as above:

Theorem

For any real-symmetric matrix polynomial $L(\lambda)$ of even degree, $l = 2m$, with invertible leading coefficient, there is a **real orthogonal** *matrix $\Theta \in \mathbb{R}^{mn \times mn}$ such that $B = A\Theta$.*

Conversely, let J, P be real canonical matrices as described, and $X = \left[\begin{array}{cccc} X_+ & X_- & V & U \end{array} \right] \in \mathbb{R}^{n \times ln}$ with $B := A\Theta$ for some real orthogonal Θ and $(XJ^{l-1}PX^T)$ nonsingular then (X, J, PX^T) is a real selfadjoint triple.

Implication for inverse problems?

The quadratic case

In the eigenvector matrix:

$$X = X_R \Pi = \begin{bmatrix} X_+ & X_- & V & U \end{bmatrix} \in \mathbb{R}^{n \times 2n}.$$

we have

$$\begin{bmatrix} X_- & V \end{bmatrix} = \begin{bmatrix} X_+ & U \end{bmatrix} \Theta$$

(began with Lancaster-Prells, 2005).

CONSTRUCTIONS

Use the $2s$ non-real e.vs. $\mu_j \pm i\nu_j$ (j=1,3,...,2s-1) to define

$$M := \text{diag}[\mu_1, \cdots, \mu_{2s-1}], \quad N := \text{diag}[\nu_1, \cdots, \nu_{2s-1}], \quad \text{in} \quad \mathbb{R}^{s\times s}.$$

and then define

$$\begin{bmatrix} M_r & N_r \\ N_r & -M_r \end{bmatrix} := \begin{bmatrix} M & -N \\ N & M \end{bmatrix}^r \begin{bmatrix} I_s & 0 \\ 0 & -I_s \end{bmatrix}.$$

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$$H_k(\Theta) := \begin{bmatrix} I_{nm} & \Theta \end{bmatrix} \begin{bmatrix} R_+^k & 0 & 0 & 0 \\ 0 & M_k & 0 & -N_k \\ 0 & 0 & -R_-^k & 0 \\ 0 & -N_k & 0 & -M_k \end{bmatrix} \begin{bmatrix} I_{nm} \\ \Theta^T \end{bmatrix}.$$

$$:= \begin{bmatrix} I_{nm} & \Theta \end{bmatrix} G_k \begin{bmatrix} I_{nm} \\ \Theta^T \end{bmatrix}; \quad \text{defines} \quad G_k \in \mathbb{R}^{ln\times ln}.$$

Cauchy to the rescue

Keeping in mind

$$H_k(\Theta) = \begin{bmatrix} I_{nm} & \Theta \end{bmatrix} G_k \begin{bmatrix} I_{nm} \\ \Theta^T \end{bmatrix}$$

and using the Cauchy interlacing inequalities we get (for the ev of $H_k(\Theta)$):

Theorem

For $k = 1, 2, \ldots$ write the (known) eigenvalues of G_k in the form

$$\lambda_1(G_k) \geq \cdots \geq \lambda_{2nm}(G_k).$$

Then for any $nm \times nm$ orthogonal matrix Θ we have

$$2\lambda_i(G_k) \geq \lambda_i(H_k(\Theta)) \geq 2\lambda_{i+nm}(G_k), \quad 1 \leq i \leq nm.$$

Semisimple, real, symmetric, QUADRATICS

We form a **real canonical triple** (X, J, PX^T) for

$$L(\lambda) = L_2\lambda^2 + L_1\lambda + L_0.$$

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We form a **real canonical triple** (X, J, PX^T) for

$$L(\lambda) = L_2\lambda^2 + L_1\lambda + L_0.$$

$$R_+ = \text{Diag}[r_1, \cdots, r_q], \quad R_- = \text{Diag}[r_{q+1}, \cdots, r_{2q}]$$

$$\beta_j = \mu_j + i\nu_j, \quad \beta_{j+1} = \bar{\beta}_j = \mu_j - i\nu_j, \quad \nu_j > 0, \quad j = 1, 3, \ldots, 2s-1.$$

$$M = \text{Diag}[\mu_1, \mu_3, \cdots, \mu_{2s-1}], \quad N = \text{Diag}[\nu_1, \nu_3, \cdots, \nu_{2s-1}]$$

$$J = \text{Diag}[R_+, R_-, \begin{bmatrix} M & -N \\ N & M \end{bmatrix}] \in \mathbb{R}^{2n\times 2n},$$

$$P = \Pi^T P_R \Pi = \text{Diag}[I_q, -I_q, -I_s, I_s] \in \mathbb{R}^{2n\times 2n},$$

$$X = \begin{bmatrix} X_+ & X_- & V & U \end{bmatrix} \in \mathbb{R}^{n\times 2n}.$$

$(X_+, X_-$ are $n \times q$ and (V, U) are $n \times s$.)

An inverse quadratic problem:

Given semisimple canonical matrices $J, P \in \mathbb{R}^{2n\times 2n}$ as above, can we always find an $X \in \mathbb{R}^{n\times 2n}$ to complete a canonical triple (X, J, PX^T)?

NO! Recall that there are $2q$ **real** eigenvalues (counting multiplicities), and s pairs of **complex conjugate** eigenvalues $0 \leq q, s \leq n$.
To characterize this problem we need another parameter: Let p be the **maximal multiplicity** of any real eigenvalue (so that $1 \leq p \leq 2n$).

Theorem

(L.-Zaballa) With the hypotheses above, there exists a canonical triple (X, J, PX^T) *if and only if*

$$q + s \geq p.$$

(Notice that $s = 0$ is possible if $q \geq p$. Also, $q + s = p$ when the system is semisimple.)

Another inverse quadratic problem

Recall that, for a real canonical triple (X, J, PX^T), $\Gamma_1 = XJPX^* = A_2^{-1}$.

Theorem

Given (semisimple) candidates J, P as components of a real canonical triple (X, J, PX^T), there is an associated semisimple, real, symmetric quadratic matrix polynomial with Γ_1 $(= XJPX^*)$ nonsingular (or $\Gamma_1 > 0$) if and only if there is an orthogonal matrix Θ such that

$$H_1(\Theta) := \begin{bmatrix} I_{nm} & \Theta \end{bmatrix} \begin{bmatrix} R_+ & 0 & 0 & 0 \\ 0 & M & 0 & -N \\ 0 & 0 & -R_- & 0 \\ 0 & -N & 0 & -M \end{bmatrix} \begin{bmatrix} I_{nm} \\ \Theta^T \end{bmatrix}$$

is nonsingular (resp. positive definite).

EPILOGUE:– GENERAL CANONICAL FORMS (L/Z)

FIRST - reduction over $\mathbb{C}$.

Primitive matrices F_j and G_j:

$$F_1 = [1], \qquad G_1 = [0],$$

$$F_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad G_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{etc..}$$

EPILOGUE:– GENERAL CANONICAL FORMS (L/Z)

$$P = \bigoplus_{j=1}^{q} \varepsilon_j F_{l_j} \bigoplus \bigoplus_{k=1}^{s} F_{2m_k},$$

$$PJ = \bigoplus_{j=1}^{q} \varepsilon_j (\alpha_j F_{l_j} + G_{l_j}) \bigoplus \bigoplus_{k=1}^{s} \begin{bmatrix} 0 & \beta_k F_{m_k} + G_{m_k} \\ \bar{\beta}_k F_{m_k} + G_{m_k} & 0 \end{bmatrix}.$$

Theorem

If $L(\lambda)$ is Hermitian and A_l is nonsingular, then there exists a selfadjoint Jordan triple of the form (X, J, PX^*). The set of numbers ε_j $(= \pm 1)$ is determined uniquely by $L(\lambda)$ up to permutation of the signs in the blocks of P corresponding to the Jordan blocks, $\alpha_j F_{l_j} + G_{l_j}$, of J with the same real eigenvalue and size.

And the Jordan form, J itself:

The numbers ε_j are, of course, ± 1 and are the *sign characteristics* of the real eigenvalues.

$$J = \bigoplus_{j=1}^{q} (\alpha_j I_{l_j} + F_{l_j}) \bigoplus \bigoplus_{k=1}^{s} \begin{bmatrix} \bar{\beta}_k I_{m_k} + F_{m_k} G_{m_k} & 0 \\ 0 & \beta_k I_{m_k} + F_{m_k} G_{m_k} \end{bmatrix},$$

and satisfies

$$J^* P = PJ,$$

i.e. J is selfadjoint in the (indefinite) P inner-product.

Epilogue: THE REAL SYMMETRIC CASE (L/Z)

Reduction over the reals is more complicated and requires another class of primitive matrices of even size, $2m$. For example, when $m = 3$

$$E_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Epilogue: THE REAL SYMMETRIC CASE (L/Z)

$$P = \bigoplus_{j=1}^{q} \varepsilon_j F_{l_j} \bigoplus \bigoplus_{k=1}^{s} F_{2m_k},$$

$$PK = \bigoplus_{j=1}^{q} \varepsilon_j (\alpha_j F_{l_j} + G_{l_j}) \bigoplus \bigoplus_{j=1}^{s} \left(\mu_j F_{2m_j} + \nu_j E_{2m_j} + \begin{bmatrix} F_{2m_j-2} & 0 \\ 0 & 0_2 \end{bmatrix} \right)$$

(Extract K from this.)

Theorem

If $L(\lambda)$ is real and symmetric and A_l is nonsingular, then there exists a real Jordan triple of the form (X_ρ, K, PX_ρ^T). The set of numbers ε_j $(= \pm 1)$ is determined uniquely by $L(\lambda)$ up to permutation of the signs in the blocks of P corresponding to the Jordan blocks of K with the same real eigenvalue and size.

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Time to go!