Second ALAMA Course on Matrix Polynomials

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Polynomials preserving the nonnegativity of matrices and NIEP

Contents

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1. NIEP (Nonnegative Inverse Eigenvalue Problem)

The **spectrum** of an n-by-n matrix A is $Spec(A) := \{\lambda_1, \ldots, \lambda_n : \lambda_i$ is an eigenvalue of $A\}$

The **spectral radius** of A is $\rho(A) := max\{|\lambda_i| : 1 \leq i \leq n\}$

NIEP: Given $\sigma = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}^n$

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Suleimanova, Perfect, Brauer, Mirsky 1950's

Ciarlet 1968, Kellogg 1971, Salzmann 1972, Fiedler 1974, Soules 1983

Loewy-London 1978, Laffey, Reams, Meehan 1995 $\longrightarrow$

Borobia 1995; Borobia-Moro-Rojo-Soto; Laffey-Šmigoc 2003 $\longrightarrow$

Boyle-Handelman 1990's; Kim-Ormes-Roush 2000, ...

2. First necessary conditions

The **moment of order** k of $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{C}$: $\quad s_k(\sigma) := \sum_{i=1}^{n} \lambda_i^k.$

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4) For $k = 1$ we have $[s_1(\sigma)]^m \leq n^{m-1} s_m(\sigma)$. Therefore

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If $s_1(\sigma) \geq 0$, then JLL $\Longrightarrow$ $s_k(\sigma) \geq 0, \quad \forall k \geq 1.$

Conclusion: The historic necessary conditions can be reduced to

$s_k(\sigma) \geq 0$ and JLL or only to JLL, if we take σ with $s_1(\sigma) \geq 0$.

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NIEP solution, case $n = 3$, Loewy-London 1978

The JLL necessary conditions are sufficient

$$\sigma = \{\lambda_1, \lambda_2, \lambda_3\} \in Spec(A_{3\times 3} \geq 0) \Longleftrightarrow \begin{cases} \sigma = \overline{\sigma} \\ \rho(\sigma) \in \sigma \\ s_1(\sigma) \geq 0 \\ (s_1(\sigma))^2 \leq 3 s_2(\sigma) \quad (JLL \;\; k = 1, m = 2) \end{cases}$$

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$(s_1(\sigma))^2 \leq 3s_2(\sigma)$ can be written: $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \leq \lambda_1^2 + \lambda_2^2 + \lambda_3^2$

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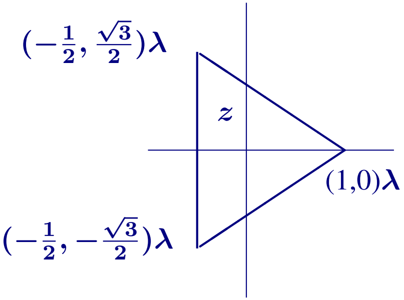
if $\sigma = \{\lambda, z, \overline{z}\}$: $(\lambda - Re(z))^2 \geq 3Im(z)^2$



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NIEP solution, case $n = 4$, real, Loewy-London 1978

$$\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4 : \lambda_i \in \mathbb{R}\} \in Spec(A_{4\times 4} \geq 0) \Longleftrightarrow \begin{cases} \rho(\sigma) \in \sigma \\ \\ s_1(\sigma) \geq 0 \end{cases}$$

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Observation: The family $\sigma = \{\sqrt{2}, \sqrt{2}, i, -i, 0, \ldots, 0\}$ satisfies $s_k \geq 0$ and JLL,

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Conclusion: The necessary conditions $\sigma_k \geq 0$ and JLL are not sufficient for no $n \geq 4$.

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$$\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \in Spec(A_{4\times 4} \geq 0,\ tr(A) = 0) \Longleftrightarrow s_1 = 0, s_2, s_3 \geq 0, s_2^2 \leq 4s_4$$

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NIEP solution, case $n = 4$, Meehan 1998, non published Ph.D. thesis

From certain refinements of the JLL conditions for the case $n = 4$.

4. Holtz necessary conditions (2005)

O. Holtz, *M-matrices satisfy Newton's inequalities*, Proc. Amer. Math. Soc. **133** (3) (2005) 711-717.

A n-by-n matrix, $A[\alpha]$ principal submatrix of A lying in the rows and columns $\alpha \subseteq \{1, \ldots, n\}$

$\sigma = \{\lambda_1, \ldots, \lambda_n\} = Spec(A)$ and $E_k(\sigma) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k} = \sum_{|\alpha|=k} \det A[\alpha]$

$$P_A(x) = \sum_{k=0}^{n} (-1)^{n-k} E_k(\sigma) x^{n-k}$$

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real diagonal matrices

diagonalizable matrices with real spectrum (the c_k are invariant under similarity)

matrices with real spectrum (the closure of the above set)

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***M*-matrices and inverse *M*-matrices (Holtz 2005)**

M-matrices := P-matrices (positive principal minors) with nondiagonal entries ≤ 0.

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If $A \geq 0$ with spectral radius $\rho(A)$, then $\rho(A)I - A$ is an M-matrix and, therefore, must verify the **Newton inequalities**

(1) $$c_k(\sigma)^2 \geq c_{k-1}(\sigma)c_{k+1}(\sigma), \quad 1 \leq k \leq n-1,$$ where

$$c_k(\sigma) = \frac{\text{degree } (n-k) \text{ coefficient of } \prod_{i=1}^{n}(x - (\rho(A) - \lambda_i))}{\binom{n}{k}}$$

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$$c_k(\sigma) = \frac{\text{degree } (n-k) \text{ coefficient of } \prod_{i=1}^{n}(x - (\rho(A) - \lambda_i))}{\binom{n}{k}}$$

Holtz conditions: if $\sigma = \{\lambda_1, \ldots, \lambda_n\} = Spec(A \geq 0)$ with spectral radius $\rho(A)$,

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4. Holtz necessary conditions (2005)

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Holtz proves that its necessary conditions and the conditions $\sigma_k \geq 0$ and JLL are mutually independent.

5. Necessary conditions on the coefficients (2007)

NIEP: Given $\boldsymbol{\sigma} = \{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n\} \subset \mathbb{C}^n$
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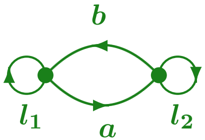
Tactics. If $P(x) = x^n + k_1 x^{n-1} + \cdots + k_j x^{n-j} + \cdots + k_n$ is realizable, we try to maximize a coefficient k_j as a function of the previous coefficients preserving the realizability for a polynomial of degree n with the same $k_1, k_2, \ldots, k_{j-1}$ as $P(x)$.

This maximum is attained.

5. Necessary conditions on the coefficients (2007)

NIEP solution, case $n = 2$

$$x^2 + k_1 x + k_2 \quad \text{is realizable} \quad \begin{pmatrix} l_1 & a \\ b & l_2 \end{pmatrix}$$



$$\Longleftrightarrow \begin{cases} a) \quad k_1 \le 0 \\ b) \quad k_2 \le \dfrac{k_1^2}{4} \end{cases}$$

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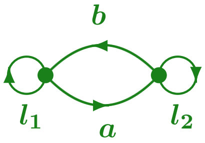
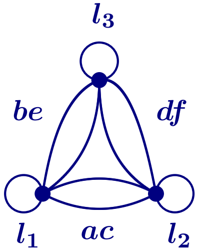


$\iff \begin{cases} a) & k_1 \leq 0 \\ b) & k_2 \leq \dfrac{k_1^2}{4} \end{cases}$

NIEP solution, case $n = 3$

$x^3 + k_1 x^2 + k_2 x + k_3$ is realizable $\begin{pmatrix} l_1 & a & b \\ c & l_2 & d \\ e & f & l_3 \end{pmatrix}$



$\iff$

$a) \quad k_1 \leq 0$

$b) \quad k_2 \leq \dfrac{k_1^2}{3}$

$$c) \quad k_3 \leq \begin{cases} \frac{k_1 k_2}{3} + \frac{2}{27}\left((k_1^2 - 3k_2)^{\frac{3}{2}} - k_1^3\right) & \text{if} \quad k_2 > -k_1^2 \\ k_1 k_2 & \text{if} \quad k_2 \leq -k_1^2 \end{cases}$$

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Necessary conditions **CC:** **If** $x^n + k_1 x^{n-1} + k_2 x^{n-2} + \cdots + k_n = P_{A\geq 0}(x)$, **then:**

a) $k_1 \leq 0$;

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Sufficient condition: Given k_1, k_2 and k_3 verifying CC $\Longrightarrow$ there exists $A \geq 0$ whose $P_A(x) = x^n + k_1x^{n-1} + k_2x^{n-2} + k_3x^{n-3} + Q(x)$, with $degree(Q(x)) \leq n-4$.

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¿$\exists \sigma$? JLL + CC $\not\Longrightarrow$ Holtz

5. Necessary conditions on the coefficients (2007)

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NIEP solution, case $n = 4$; NIEP solution, cases trace 0 more general

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If $\sigma = \{\lambda_1, \dots, \lambda_n\}$ is self-conjugate, then are equivalent:

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These conditions **are not sufficient:**

$\{1, 1, -\frac{3\sqrt{3}}{8} \pm \frac{3}{8} i\}$ satisfies $s_1 \geq 0$ + JLL + Holtz + CC, but is not realizable.

6. Polynomial matrix functions. Interpolation

Let $p(t) = a_m t^m + a_{m-1} t^{m-1} + \dots a_1 t + a_0$, with $a_i \in \mathbb{C}$, and A n-by-n matrix.

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If $A = S \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} S^{-1} \Longrightarrow p(A) = S \begin{pmatrix} p(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & p(\lambda_n) \end{pmatrix} S^{-1}.$

6. Polynomial matrix functions. Interpolation

If A is not diagonalizable and has Jordan c.f. $A = S \begin{pmatrix} J_{n_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{n_s}(\lambda_s) \end{pmatrix} S^{-1}$, then

$$p(A) = S \begin{pmatrix} p(J_{n_1}(\lambda_1)) & & 0 \\ & \ddots & \\ 0 & & p(J_{n_s}(\lambda_s)) \end{pmatrix} S^{-1} \qquad (2)$$

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If A is not diagonalizable and has Jordan c.f. $A = S \begin{pmatrix} J_{n_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{n_s}(\lambda_s) \end{pmatrix} S^{-1}$, then

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Theorem: Let $A \in M_n$ be given and $p(t)$ and $r(t)$ be given polynomials.
If $m_A(t) = (t - \lambda_1)^{r_1} \dots (t - \lambda_\mu)^{r_\mu}$ is m.p. of A, then

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Lagrange's formula (multiplicity 1); Lagrange-Hermitte's formula (multiplicity > 1)

7. Nonpolynomial matrix functions

A. Horn, C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1994.

If $f(t)$ is a continuous scalar function, $t \in \mathbb{R}, \mathbb{C}$ how can $f(A)$ be defined with $A \in M_n$?

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$$\text{as } \lim_{\epsilon \to 0} A_\epsilon = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = J_2(\lambda), \text{ if } f(A) \text{ is continuous of } J_2(\lambda) \Longrightarrow$$

$$f(J_2(\lambda)) = f(\lim_{\epsilon \to 0} A_\epsilon) = \lim_{\epsilon \to 0} f(A_\epsilon) = \begin{pmatrix} f(\lambda) & f'(\lambda) \\ 0 & f(\lambda) \end{pmatrix}$$

$$\text{If } J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \implies f(J_k(\lambda)) = \begin{pmatrix} f(\lambda) & f'(\lambda) & \frac{1}{2}f''(\lambda) & \cdots & \cdots & \frac{1}{(k-1)!}f^{(k-1)}(\lambda) \\ 0 & f(\lambda) & f'(\lambda) & \ddots & \cdots & \cdots \\ 0 & 0 & f(\lambda) & \ddots & \ddots & \cdots \\ \vdots & \vdots & \cdots & \ddots & \ddots & \frac{1}{2}f''(\lambda) \\ \vdots & \vdots & \cdots & & & f'(\lambda) \\ \vdots & \vdots & \cdots & & & f(\lambda) \end{pmatrix} \tag{5}$$

If $f(A)$ continuous of A on the closure of $\{\text{diagonalizable } A \in M_n\}$ whose spectra lie in a compact $K \implies f(t)$ is analytic in an open $D \supset K$.
If $f(A)$ continuous on all $M_n \implies f(t)$ must be entire.

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Primary matrix function $f(A)$ associated with a scalar function $f(t)$.

$$A \in M_n, \text{ m.p. } m_A(t) = (t-\lambda_1)^{r_1} \dots (t-\lambda_\mu)^{r_\mu} \text{ Jcf } A = S \begin{pmatrix} J_{n_1}(\lambda_{\mu_1}) & & 0 \\ & \ddots & \\ 0 & & J_{n_s}(\lambda_{\mu_s}) \end{pmatrix} S^{-1}$$

$f(t)$ s.f. with $Spec(A) \in int(dom(f))$ and $(r_i - 1)$-differentiable of λ_i.

$$\text{Then, define } f(A) := S \begin{pmatrix} f(J_{n_1}(\lambda_{\mu_1})) & & 0 \\ & \ddots & \\ 0 & & f(J_{n_s}(\lambda_{\mu_s})) \end{pmatrix} S^{-1}, \text{ with } f(J_k(\lambda)) \text{ as in (5).}$$

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Known cases: If $f(t) = p(t)$ is polynomial $\Longrightarrow$ the associated pmf is $f(A) = p(A), \ \forall A$.

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Intermediate case: $f(t)$ analytic with p.s. $f(t) = a_0 + a_1 t + a_2 t^2 + \ldots$ with r.c. $R > 0$. $A \in M_n$ with $\rho(A) < R$, then the matrix p.s. $f(A) \equiv a_0 I + a_1 A + a_2 A^2 + \ldots$ converges ($\forall$ norm) and its sum is the pmf associated with $f(t)$.

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General case: A y $f(t)$ under the conditions of the pmf. Then

a) There exists a polynomial $r(t)$ of degree $n-1$ such that $f(A) = r(A)$.
   $r(t)$ can be taken to be any polynomial that interpolates $f(t)$ and its derivatives at the roots of $m_A(t)$.

b) If A has J.c.f. $J_{n_1}(\lambda_{\mu_1}) \oplus \cdots \oplus J_{n_s}(\lambda_{\mu_s})$, then
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Remark: The pmf $f(A)$ is a polynomial function at A, but depends of A.

8. Functions preserving nonnegativity of matrices

G. Bharali, O. Holtz, *Functions preserving nonnegativity of matrices*,
SIAM J. Matrix Anal. Appl. **30** (2008) no.1, 84-101

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$$\mathcal{F}_n := \{f \;\; entire : \;\; A \in \mathbb{R}^{n \times n}, \;\; A \geq 0 \Longrightarrow f(A) \geq 0\}$$

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Proof: $f(diag(A_n, 0)) = diag(f(A_n), 0) \Longrightarrow f(A_n) \geq 0 \Longrightarrow f \in \mathcal{F}_n$

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Proof: $If\ A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \Longrightarrow f(A) = \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-2} & a_{n-1} \\ 0 & a_0 & a_1 & \ldots & a_{n-3} & a_{n-2} \\ 0 & 0 & a_0 & \ldots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a_0 & a_1 \\ 0 & 0 & 0 & \ldots & 0 & a_0 \end{pmatrix}$

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Remark. Condition of Proposition 2 is not sufficient:

$$f(x) = -x^n + \sum_{j=0}^{n-1} a_j x^j \;\text{ with }\; a_j \geq 0 \quad \text{do not preserve nonnegativity.}$$

Proof: $\exists x_0 \geq 0 \quad s.t. \quad f(x) < 0 \quad \forall x \in (x_0, \infty)$.

Then $A = rI_n \geq 0$ for some $r \in (x_0, \infty)$ while $f(A)$ has negative diagonal entries.

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5. Preserving nonnegativity of (block) triangular matrices T. The *divided difference* [8] of a smooth function f at points $x_1, \ldots, x_n$ is defined via the recurrence relation

$$f[x_1, \ldots, x_k] := \begin{cases} \dfrac{f[x_2, \ldots, x_k] - f[x_1, \ldots, x_{k-1}]}{x_k - x_1}, & x_1 \neq x_k, \\ f^{(k-1)}(x_1)/(k-1)!, & x_1 = x_k \end{cases} \quad \textit{where} \;\; f[x] := f(x)$$

THEOREM 6. $f \in \mathcal{F}_n(T) \iff f[x_1, \ldots, x_k] \geq 0, \;\; x_1, \ldots, x_k \geq 0, \;\; k = 1, \ldots, n \quad (1)$
or, equivalently, that all derivatives of order up to $n-1$ *are nonnegative on* $\mathbb{R}_+$.

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$$f[x_1, \ldots, x_k] := \begin{cases} \dfrac{f[x_2, \ldots, x_k] - f[x_1, \ldots, x_{k-1}]}{x_k - x_1}, & x_1 \neq x_k, \\ f^{(k-1)}(x_1)/(k-1)!, & x_1 = x_k \end{cases} \quad where \quad f[x] := f(x)$$

THEOREM 6. $f \in \mathcal{F}_n(T) \iff f[x_1, \ldots, x_k] \geq 0, \quad x_1, \ldots, x_k \geq 0, \quad k = 1, \ldots, n \quad (1)$
or, equivalently, that all derivatives of order up to $n-1$ *are nonnegative on* $\mathbb{R}_+$.

PROPOSITION 8. *Let* f *be an entire function, and let* $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}, \quad \sigma(A) \cap \sigma(C) = \emptyset.$
Then $f(M) = \begin{pmatrix} f(A) & f(A)X - Xf(C) \\ 0 & f(C) \end{pmatrix}$, *where* X *is the solution to* $AX - XC = B$.

COROLLARY 9. $f \in \mathcal{F}_n\left(\begin{pmatrix} A_p & B \\ 0 & C_q \end{pmatrix}\right) \iff f \in \mathcal{F}_{max\{p,q\}}$ *and* $f(A)X - Xf(C) \geq 0$.

Proof: The matrices $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \geq 0, \quad \sigma(A) \cap \sigma(C) = \emptyset$ are dense in the set of all block upper-triangular matrices.

The above proposition does not allow for an explicit formula as in theorem 6.

6. Preserving nonnegativity of circulant matrices C_n. A circulant matrix A is determined by its first row $(a_0, \ldots, a_{n-1})$

All circulant matrices of size n are polynomials in the basic circulant matrix C_n which implies in particular that any function $f(A)$ of a circulant matrix is a circulant matrix as well. Moreover, the eigenvalues of a circulant matrix are determined by its first row $[a_0, a_1, \ldots, a_{n-1}]$ by the formula

$$\left\{ \sum_{j=0}^{n-1} w^{jk} a_j : \quad k = 0, \ldots, n-1 \right\}, \quad \text{where} \ \ w = e^{2\pi i/n}.$$

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Thus, the elements $(f_0, \ldots, f_{n-1})$ of the first row of $f(A)$ can be read off from its spectrum:

$$f_l = \frac{1}{n} \sum_{k=0}^{n-1} w^{-lk} f\left(\sum_{j=0}^{n-1} w^{jk} a_j \right), \quad l = 0, \ldots, n-1.$$

THEOREM 10. $f \in \mathcal{F}_n(C_n) \iff$ *for* $l = 0, \ldots, n-1$,

$$\sum_{k=0}^{n-1} w^{-lk} f\left(\sum_{j=0}^{n-1} w^{jk} a_j \right) \geq 0 \quad \textit{whenever} \;\; a_j \geq 0, \;\; j = 0, \ldots, n-1, \;\; \textit{where} \;\; w = e^{2\pi i/n}.$$

7. Characterization of $\mathcal{F}_n$ for small values of n.

7.1 The case $n = 1$. $f \in \mathcal{F}_1 \Longleftrightarrow f(x) \geq 0 \ \ \forall x \geq 0$. An alternative characterization:

PROPOSITION 11. *A function f having finitely many zeros is in $\mathcal{F}_1$ if and only if it has the form*

$$f(z) = g(z) \prod_{\alpha,\beta} ((z+\alpha)^2 + \beta^2) \prod_{\gamma} (z+\gamma), \quad \alpha, \beta \in \mathbb{R}, \quad \gamma \geq 0$$

and g is an entire function that has no zeros in $\mathbb{C}$ and is positive on $\mathbb{R}_+$.

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7.2 The case $n = 2$. The specialization of the Lemmas 4 and 5 to the case $n = 2$ gives the following corollary.

COROLLARY 12. $f \in \mathcal{F}_2 \iff f\left(\begin{pmatrix} a & b \\ b & c \end{pmatrix} > 0\right) \geq 0.$

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THEOREM 13. *An entire function f is in $\mathcal{F}_2$ if and only it satisfies the conditions*

$$(6) \qquad f(x+y) - f(x-y) \geq 0 \quad \forall x, y \geq 0$$

$$(7) \qquad (x+y-z)f(x-y) + (z-x+y)f(x+y) \geq 0 \ \ \forall x \geq z \geq 0, \ y \geq x-z,$$

or, equivalently, $(8) \qquad (x+y)f(x-y) + (y-x)f(x+y) \geq 0 \quad \forall y \geq x \geq 0.$

8. Preserving nonnegative symmetric matrices S.

8.1 Preserving nonnegative definite nonnegative symmetric matrices $S >> 0$.

RESULT 14 . *An entire function* f *preserves the class* $S >> 0 \Longleftrightarrow$

(1) $$f[x_1, \ldots, x_k] \geq 0, \quad x_1, \ldots, x_k \geq 0, \; k = 1, \ldots, n$$

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(1) is not sufficient for a function to preserve nonnegativity of all nonnegative symmetric matrices.

$f(x) = 1 + x + \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4$ satisfies (1) with $n = 2$, but it maps $\begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix}$ (*not nonnegative definite*) to a matrix with negative off-diagonal entries for any $M > \sqrt{3/2}$.

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8.2 Even and odd functions preserving nonnegativity of symmetric matrices S.

THEOREM 17. *An even entire function* $f(z) =: g(z^2) \in \mathcal{F}_n(S) \iff g$ *satisfies (1).*

An odd entire function $f(z) =: zh(z^2) \in \mathcal{F}_n(S) \iff h$ *satisfies (1).*

Proof: $A \geq 0$ symmetric $\Longrightarrow A^2 \geq 0$ symmetric and nonnegative definite $\Longrightarrow g(A^2) \geq 0$.

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1) If $p(x) = k_0 + k_1 x + \cdots + k_m x^m \in \mathcal{P}_n \Longrightarrow k_0, k_1, \ldots, k_{n-1} \geq 0$.

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and if $p(x) = k_0 + k_1 x + k_2 x^2 + \cdots + k_{2n+1} x^{2n+1}$ we have

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4) If $p(x) = k_0 + k_1 x + \cdots + k_m x^m \in \mathcal{P}_2$, then

i) $k_0 > 0, \ k_1 > 0$ (the first nonzero even (and odd) coefficient must be positive),

ii) If m even: $k_m > 0$ and $k_{m-1} > 0$ (the last nonzero odd coefficient must be positive).

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If $\sigma = Spec(A \geq 0)$ and $p(x) \in \mathcal{P}_n$, then $p(\sigma) = Spec(p(A) \geq 0)$ must verify the necessary conditions for the NIEP: $\forall k, m \geq 1$

$$s_k(\sigma) \geq 0 \longrightarrow s_k(p(\sigma)) \geq 0$$

JLL $[s_k(\sigma)]^m \leq n^{m-1} s_{km}(\sigma) \longrightarrow [s_k(p(\sigma))]^m \leq n^{m-1} s_{km}(p(\sigma))$

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J.P. D'Angelo, *Inequalities from Complex Analysis*, The Carus Mathematical Monographs, The Mathematical Association of America, 2002.

$p(x) > 0, \ \forall x \geq 0 \Longleftrightarrow (1+x)^n p(x) = r(x),$ with $n \in \mathbb{N}$ and $r(x)$ with positive coef.

$p(x) \geq 0, \ \forall x \Longleftrightarrow p(x)$ is a sum of square polynomials.

To characterize $\mathcal{P}_n = \{p(x) : p(A) \geq 0, \ \forall A_{n \times n} \geq 0\}$ in D'Angelo terms.

¡Thank you

for your attention!