Finite and infinite structure of matrix polynomials: local and global approaches

2nd ALAMA Courses on Matrix Polynomials

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Outline

Outline

Let $\mathcal{D}$ be a PID.
$\mathrm{Gl}_m(\mathcal{D})$: set of invertible matrices in $\mathcal{D}^{m\times m}$

Definition

Two matrices $D_1, D_2 \in \mathcal{D}^{m\times n}$ are **equivalent in** $\mathcal{D}$ *if there exist $U \in \mathrm{Gl}_m(\mathcal{D})$ and $V \in \mathrm{Gl}_n(\mathcal{D})$ such that $D_2 = UD_1V$.*

Theorem (Normal Smith form in $\mathcal{D}$)

Every matrix $D \in \mathcal{D}^{m\times n}$ is equivalent in $\mathcal{D}$ to a matrix of the form

$$\begin{bmatrix} \mathrm{Diag}(\alpha_1, \ldots, \alpha_r) & 0 \\ 0 & 0 \end{bmatrix}$$

where $r = \mathrm{rank}\, D$, $\alpha_1, \ldots, \alpha_r$ are nonzero elements of $\mathcal{D}$ (unique up to the product by units) and $\alpha_1 \mid \cdots \mid \alpha_r$.

Remark: α_1 is the greatest common divisor of the entries of D (unique up to the product by units).

Let $\mathcal{F}$ be the field of fractions of $\mathcal{D}$, that is, $\mathcal{F} = \left\{ \frac{f}{g} : f, g \in \mathcal{D}, g \neq 0 \right\}$.

Definition

Two matrices $F_1, F_2 \in \mathcal{F}^{p \times m}$ are **equivalent in** $\mathcal{D}$ *if there exist $U \in \mathrm{Gl}_m(\mathcal{D})$, $V \in \mathrm{Gl}_n(\mathcal{D})$ such that $F_2 = UF_1V$.*

Theorem (Smith-McMillan form in $\mathcal{D}$)

Every matrix $F \in \mathcal{F}^{m \times n}$ is equivalent in $\mathcal{D}$ to a matrix of the form

$$\begin{bmatrix} \mathrm{Diag}\left(\frac{\epsilon_1}{\psi_1}, \ldots, \frac{\epsilon_r}{\psi_r}\right) & 0 \\ 0 & 0 \end{bmatrix}$$

where $r = \operatorname{rank} F$, ϵ_i, ψ_i are nonzero coprime elements of $\mathcal{D}$ (unique up to the product by units) and $\epsilon_1 \mid \cdots \mid \epsilon_r$ while $\psi_r \mid \cdots \mid \psi_1$.

Theorem (Smith-McMillan form in $\mathcal{D}$)

Every matrix $F \in \mathcal{F}^{m\times n}$ is equivalent in $\mathcal{D}$ to a matrix of the form

$$\begin{bmatrix} \mathrm{Diag}\left(\frac{\epsilon_1}{\psi_1}, \dots, \frac{\epsilon_r}{\psi_r}\right) & 0 \\ 0 & 0 \end{bmatrix}$$

where $r = \operatorname{rank} F$, ϵ_i, ψ_i are nonzero coprime elements of $\mathcal{D}$ (unique up to the product by units) and $\epsilon_1 \mid \cdots \mid \epsilon_r$ while $\psi_r \mid \cdots \mid \psi_1$.

Proof.- Let $F = (f_{ij}) \in \mathcal{F}^{m\times n}$, $f_{ij} = \frac{a_{ij}}{b_{ij}}$, $a_{ij}, b_{ij} \in \mathcal{D}$, $b_{ij} \neq 0$.
Let b be the least common multiple (unique up to product by units) of b_{ij}.
Let $D = bF$. Then $D \in \mathcal{D}^{m\times n}$.
Take D to Smith normal form, that is, there exist $U \in \mathrm{Gl}_m(\mathcal{D})$, $V \in \mathrm{Gl}_n(\mathcal{D})$ so that

$$UDV = \begin{bmatrix} \mathrm{Diag}(\alpha_1, \dots, \alpha_r) & 0 \\ 0 & 0 \end{bmatrix}$$

where $r = \operatorname{rank} D$, $\alpha_1, \dots, \alpha_r$ are nonzero elements of $\mathcal{D}$ (unique up to the product by units) and $\alpha_1 \mid \cdots \mid \alpha_r$.
Divide by b and cancel common factors so $\frac{\alpha_i}{b} = \frac{\epsilon_i}{\psi_i}$:

$$UFV = \begin{bmatrix} \mathrm{Diag}(\frac{\epsilon_1}{\psi_1}, \dots, \frac{\epsilon_r}{\psi_r}) & 0 \\ 0 & 0 \end{bmatrix}.$$

Remark: $\psi_1 = b$ (up to the product by a unit).

Theorem (Smith-McMillan form in $\mathcal{D}$)

Every matrix $F \in \mathcal{F}^{m\times n}$ is equivalent in $\mathcal{D}$ to a matrix of the form

$$\begin{bmatrix} \mathrm{Diag}\left(\frac{\epsilon_1}{\psi_1}, \ldots, \frac{\epsilon_r}{\psi_r}\right) & 0 \\ 0 & 0 \end{bmatrix}$$

where $r = \mathrm{rank}\, F$, ϵ_i, ψ_i are nonzero coprime elements of $\mathcal{D}$ (unique up to the product by units) and $\epsilon_1 \mid \cdots \mid \epsilon_r$ while $\psi_r \mid \cdots \mid \psi_1$.

- When $\mathcal{D}$ is the ring of polynomials $\rightarrow$ global structure
- When $\mathcal{D}$ is the local ring at a prime polynomial $\rightarrow$ local structure
- When $\mathcal{D}$ is the ring of proper rational functions $\rightarrow$ infinite structure

$\mathbb{F}$ any arbitrary field

$$P(s) = \begin{bmatrix} p_{11}(s) & \cdots & p_{1n}(s) \\ \vdots & \ddots & \vdots \\ p_{m1}(s) & \cdots & p_{mn}(s) \end{bmatrix} \in \mathbb{F}[s]^{m\times n} \quad \text{polynomial matrix}$$

$$P(s) = P_d s^d + P_{d-1}s^{d-1} + \cdots + P_0 \in \mathbb{F}^{m\times n}[s] \quad \text{matrix polynomial}$$

Example:

$$\begin{bmatrix} s^3+3s^2 & s^2+2s+1 \\ 1 & 2s^3+3s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} s^3 + \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} s + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The degree of $P(s)$, $d(P(s))$, is the degree of the polynomial of highest degree or the degree of the matrix polynomial.

Outline

$\mathbb{F}[s]$ is a PID.

$\mathrm{Gl}_m(\mathbb{F}[s])$: set of invertible matrices in $\mathbb{F}[s]^{m\times m}$ (unimodular matrices).

$$U(s) \in \mathrm{Gl}_m(\mathbb{F}[s]) \Leftrightarrow U(s) \in \mathbb{F}[s]^{m\times m} \text{ and } \det U(s) = c \neq 0.$$

Definition

$P_1(s), P_2(s) \in \mathbb{F}[s]^{m\times n}$ are **global/finite equivalent** *or* **equivalent in $\mathbb{F}[s]$** *if there exist $U(s) \in \mathrm{Gl}_m(\mathbb{F}[s])$, $V(s) \in \mathrm{Gl}_n(\mathbb{F}[s])$ so that*

$$P_2(s) = U(s)P_1(s)V(s).$$

Theorem (Global/Finite Smith normal form)

Every $P(s) \in \mathbb{F}[s]^{m\times n}$ is global/finite equivalent to a matrix of the form

$$\begin{bmatrix} \mathrm{Diag}(\alpha_1(s), \dots, \alpha_r(s)) & 0 \\ 0 & 0 \end{bmatrix}$$

where $r = \operatorname{rank} P(s)$ and $\alpha_1(s), \dots, \alpha_r(s)$ are nonzero monic polynomials such that $\alpha_1(s) \mid \cdots \mid \alpha_r(s)$.

Definition

$\alpha_1(s), \dots, \alpha_r(s)$ are the **global/finite invariant factors** *of $P(s)$.*

Let $P(s) \in \mathbb{F}[s]^{m\times n}$ and $\alpha_1(s) \mid \ldots \mid \alpha_r(s)$ be its global/finite invariant factors. Decompose them into monic irreducible factors over $\mathbb{F}$:

$$\alpha_1(s) = \pi_1(s)^{d_{11}} \cdots \pi_t(s)^{d_{1t}},$$

$$\alpha_2(s) = \pi_1(s)^{d_{21}} \cdots \pi_t(s)^{d_{2t}},$$

$$\vdots$$

$$\alpha_r(s) = \pi_1(s)^{d_{r1}} \cdots \pi_t(s)^{d_{rt}},$$

with $d_{rj} \geq \cdots \geq d_{2j} \geq d_{1j} \geq 0$, $j = 1, \ldots t$.

Definition

The powers $\pi_j(s)^{d_{ij}}$ with $d_{ij} > 0$ are the **(finite) elementary divisors** *of $P(s)$ in $\mathbb{F}$.*

Definition

The roots of the global invariant factors (or of the elementary divisors) in the algebraic closure of $\mathbb{F}$ are the **finite zeros** *of $P(s)$.*

Let $\mathbb{F}(s)$ be the field of rational functions, that is,

$$\mathbb{F}(s) = \left\{ \frac{p(s)}{q(s)} : p(s), q(s) \in \mathbb{F}[s], q(s) \neq 0 \right\}.$$

Definition

$R_1(s), R_2(s) \in \mathbb{F}(s)^{m\times n}$ are **global/finite equivalent** *or* **equivalent in $\mathbb{F}[s]$** *if there exist $U(s) \in \mathrm{Gl}_m(\mathbb{F}[s])$, $V(s) \in \mathrm{Gl}_n(\mathbb{F}[s])$ so that*

$$R_2(s) = U(s)R_1(s)V(s).$$

Theorem (Global/finite Smith-McMillan form)

Every $R(s) \in \mathbb{F}(s)^{m\times n}$ is global/finite equivalent to a matrix of the form

$$\begin{bmatrix} \mathrm{Diag}\left(\frac{\epsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\epsilon_r(s)}{\psi_r(s)}\right) & 0 \\ 0 & 0 \end{bmatrix}$$

where $r = \operatorname{rank} R(s)$, $\epsilon_1(s), \ldots, \epsilon_r(s), \psi_1(s), \ldots, \psi_r(s) \in \mathbb{F}[s]$ are nonzero monic, $\epsilon_i(s), \psi_i(s)$ are coprime for each i, and $\epsilon_1(s) \mid \cdots \mid \epsilon_r(s)$ while $\psi_r(s) \mid \cdots \mid \psi_1(s)$.

Remark: $\psi_1(s)$ is the monic least common denominator of $R(s)$.

Theorem (Global/finite Smith-McMillan form)

Every $R(s) \in \mathbb{F}(s)^{m\times n}$ is global/finite equivalent to a matrix of the form

$$\begin{bmatrix} \mathrm{Diag}\left(\frac{\epsilon_1(s)}{\psi_1(s)}, \dots, \frac{\epsilon_r(s)}{\psi_r(s)}\right) & 0 \\ 0 & 0 \end{bmatrix}$$

where $r = \operatorname{rank} R(s)$, $\epsilon_1(s), \dots, \epsilon_r(s), \psi_1(s), \dots, \psi_r(s) \in \mathbb{F}[s]$ are nonzero monic, $\epsilon_i(s), \psi_i(s)$ are coprime for each i, and $\epsilon_1(s) \mid \cdots \mid \epsilon_r(s)$ while $\psi_r(s) \mid \cdots \mid \psi_1(s)$.

Definition

$\frac{\epsilon_1(s)}{\psi_1(s)}, \dots, \frac{\epsilon_r(s)}{\psi_r(s)}$ *are the* **global/finite invariant rational functions** *of $R(s)$.*

Definition

The roots of $\epsilon_1(s), \dots, \epsilon_r(s)$ in the algebraic closure of $\mathbb{F}$ are the **finite zeros** *of $R(s)$.*
The roots of $\psi_1(s), \dots, \psi_r(s)$ in the algebraic closure of $\mathbb{F}$ are the **finite poles** *of $R(s)$.*

Remark

Notice that $R(s)$ is a polynomial matrix $\Leftrightarrow \psi_1(s) = 1$. Therefore,

a polynomial matrix does not have finite poles.

Example:

$\mathbb{F} = \mathbb{R}$

$$R(s) = \begin{bmatrix} \dfrac{s+1}{s} & -\dfrac{s}{s+1} \\ 1 & \dfrac{1}{(s+1)^2} \end{bmatrix} \in \mathbb{R}(s)^{2\times 2}$$

Monic least common denominator of $R(s)$: $s(s+1)^2$

$$P(s) = s(s+1)^2 R(s) = \begin{bmatrix} (s+1)^3 & -s^2(s+1) \\ s(s+1)^2 & s \end{bmatrix} \in \mathbb{F}[s]^{2\times 2}$$

The global Smith form of $P(s) = s(s+1)^2 R(s)$ is

$$\begin{bmatrix} 1 & 0 \\ 0 & s(s+1)^3(s^2+1) \end{bmatrix}.$$

The finite elementary divisors of $P(s)$ in $\mathbb{R}$ are $s, (s+1)^3, s^2+1$.

The global Smith-McMillan form of $R(s)$ is

$$\begin{bmatrix} \dfrac{1}{s(s+1)^2} & 0 \\ 0 & (s+1)(s^2+1) \end{bmatrix}.$$

Outline

⋆ A ring with exactly one maximal ideal is called a **local ring**.

⋆ If $\mathcal{R}$ is a ring and there exists an ideal $I \neq \mathcal{R}$ such that every $r \in \mathcal{R} \setminus I$ is a unit in $\mathcal{R}$ then $\mathcal{R}$ is a local ring and I its maximal ideal.

⋆ Let P be a prime ideal of $\mathcal{R}$. Then $S = \mathcal{R} \setminus P$ is a multiplicatively closed set. $S^{-1}\mathcal{R} = \left\{\frac{r}{s} : r \in \mathcal{R}, s \in S\right\}$ is a local ring ($S^{-1}P$ is its maximal ideal).

⋆ The prime ideals of $\mathbb{F}[s]$ are $(\pi(s))$ with $\pi(s)$ a prime polynomial.
Define $S = \mathbb{F}[s] \setminus (\pi(s))$. Denote $S^{-1}\mathbb{F}[s]$ as $\mathbb{F}_\pi(s)$:

$$\mathbb{F}_\pi(s) = S^{-1}\mathbb{F}[s] = \left\{\frac{p(s)}{q(s)} \in \mathbb{F}(s) : \gcd(q(s), \pi(s)) = c\right\}$$

is a local ring: the **local ring of** $\mathbb{F}[s]$ **at** $\pi(s)$.

⋆ The units of $\mathbb{F}_\pi(s)$ are the rational functions with numerator and denominator both prime with $\pi(s)$.
⋆ Moreover,

$$\frac{p(s)}{q(s)} \in \mathbb{F}_\pi(s) \Rightarrow \gcd(q(s), \pi(s)) = c \Rightarrow \frac{p(s)}{q(s)} = \frac{p_1(s)}{q(s)}\pi(s)^d = u_\pi(s)\pi(s)^d$$

with $p(s) = p_1(s)\pi(s)^d$, $\gcd(p_1(s), \pi(s)) = c$, $d \geq 0$ and $u_\pi(s) = \frac{p_1(s)}{q(s)}$ a unit of $\mathbb{F}_\pi(s)$.

$$\mathbb{F}_\pi(s) = \left\{u_\pi(s)\pi(s)^d : u_\pi(s) \text{ a unit}, d \in \mathbb{N} \cup \{0\}\right\} \cup \{0\}.$$

$\pi(s) \in \mathbb{F}[s]$ prime.

$$\mathbb{F}_\pi(s) = \left\{ \frac{p(s)}{q(s)} \in \mathbb{F}(s) : \gcd(q(s), \pi(s)) = c \right\} = \left\{ u_\pi(s)\pi(s)^d : u_\pi(s) \text{ a unit}, d \geq 0 \right\} \cup \{0\}$$

⋆ Units ($u_\pi(s)$): rational functions with both numerat. and denominat. prime with $\pi(s)$.

⋆ Notice that $u_{\pi 1}(s)\pi(s)^{d_1} \mid u_{\pi 2}(s)\pi(s)^{d_2} \Leftrightarrow d_1 \leq d_2$
$(u_{\pi_2}(s)\pi(s)^{d_2} = u_{\pi_2}(s)u_{\pi_1}(s)^{-1}\pi(s)^{d_2-d_1}u_{\pi_1}(s)\pi(s)^{d_1})$

⋆ $\mathbb{F}_\pi(s)$ is a PID and its field of fractions is $\mathbb{F}(s)$:
$\frac{u_{\pi 1}(s)\pi(s)^{d_1}}{u_{\pi 2}(s)\pi(s)^{d_2}} \in \mathbb{F}(s)$; $\frac{p(s)}{q(s)} = \frac{p_1(s)\pi(s)^{d_1}}{q_1(s)\pi(s)^{d_2}}$, $p_1(s), q_1(s)$ units ($d_1 = 0$ or $d_2 = 0$).

⋆ $\mathrm{Gl}_m(\mathbb{F}_\pi(s))$: set of invertible matrices in $\mathbb{F}_\pi(s)^{m\times m}$.

$$U(s) \in \mathrm{Gl}_m(\mathbb{F}_\pi(s)) \Leftrightarrow U(s) \in \mathbb{F}_\pi(s)^{m\times m} \text{ and } \det U(s) = u_\pi(s).$$

Definition

$R_1(s), R_2(s) \in \mathbb{F}(s)^{m\times n}$ are **local equivalent at** $\pi(s)$ *or* **equivalent in** $\mathbb{F}_\pi(s)$ *if there exist $U(s) \in \mathrm{Gl}_m(\mathbb{F}_\pi(s))$, $V(s) \in \mathrm{Gl}_n(\mathbb{F}_\pi(s))$ so that*

$$R_2(s) = U(s)R_1(s)V(s).$$

$\pi(s) \in \mathbb{F}[s]$ prime. $\mathbb{F}_\pi(s) = \left\{ u_\pi(s)\pi(s)^d : u_\pi(s) \text{ a unit}, d \geq 0 \right\} \cup \{0\}$.

Theorem (Local Smith-McMillan form)

Every $R(s) \in \mathbb{F}(s)^{m\times n}$ is local equivalent at $\pi(s)$ to a matrix of the form

$$\begin{bmatrix} \mathrm{Diag}\left(\frac{1}{\pi(s)^{-h_1}}, \ldots, \frac{1}{\pi(s)^{-h_k}}, 1 \ldots, 1, \frac{\pi(s)^{h_u}}{1}, \ldots, \frac{\pi(s)^{h_r}}{1}\right) & 0 \\ 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} \mathrm{Diag}(\pi(s)^{h_1}, \ldots, \pi(s)^{h_r}) & 0 \\ 0 & 0 \end{bmatrix}$$

where $r = \mathrm{rank}\, R(s)$ and
$h_1 \leq \cdots \leq h_k < 0 = h_{k+1} = \cdots = h_{u-1} < h_u \leq \cdots \leq h_r$ *are integers.*

Definition

$\pi(s)^{h_1}, \ldots, \pi(s)^{h_r}$ *are the* **local invariant rational functions at** $\pi(s)$ *of* $R(s)$.
$h_1, \ldots, h_r$ *are the* **invariant orders at** $\pi(s)$ *of* $R(s)$.
$h_u, \ldots, h_r$ *are the orders of the* **zeros at** $\pi(s)$ *of* $R(s)$.
$-h_1, \ldots, -h_k$ *are the orders of the* **poles at** $\pi(s)$ *of* $R(s)$.

Remark

$R(s) \in \mathbb{F}_\pi(s)^{m\times n} \Leftrightarrow h_1, \ldots, h_r$ *are nonnegative integers* $\Leftrightarrow R(s)$ *has no poles at* $\pi(s)$.

Relation between the global and local structures of polynomial matrices:

Let $P(s) \in \mathbb{F}[s]^{m\times n}$.
Let $\alpha_1(s), \ldots, \alpha_r(s)$ be its global invariant factors.

Let $\pi(s)$ be a monic prime polynomial.

Write $\alpha_i(s) = \gamma_i(s)\pi(s)^{d_i}$, $\gcd(\gamma_i(s), \pi(s)) = 1$, $d_i \geq 0$.

Notice that $\pi(s)^{d_i}$ with $d_i > 0$ are the elementary divisors powers of $\pi(s)$ of $P(s)$.

There exist $U(s) \in \mathrm{Gl}_m(\mathbb{F}[s])$, $V(s) \in \mathrm{Gl}_n(\mathbb{F}[s])$ such that

$$P(s) = U(s) \begin{bmatrix} \mathrm{Diag}\,(\alpha_1(s), \ldots, \alpha_r(s)) & 0 \\ 0 & 0 \end{bmatrix} V(s) =$$

$$\underbrace{U(s) \begin{bmatrix} \mathrm{Diag}\,(\gamma_1(s), \ldots, \gamma_r(s)) & 0 \\ 0 & I_{m-r} \end{bmatrix}}_{\in \mathrm{Gl}_m(\mathbb{F}_\pi(s))} \begin{bmatrix} \mathrm{Diag}\,\left(\pi(s)^{d_1}, \ldots, \pi(s)^{d_r}\right) & 0 \\ 0 & 0 \end{bmatrix} \underbrace{V(s)}_{\in \mathrm{Gl}_n(\mathbb{F}_\pi(s))}.$$

Therefore, $\pi(s)^{d_1}, \ldots, \pi(s)^{d_r}$ are the local invariant rational functions at $\pi(s)$ of $P(s)$.

Hence, the elementary divisors powers of $\pi(s)$ are the nontrivial local invariant rational functions at $\pi(s)$.

Equivalence is a local property in $\mathbb{F}[s]$:

$$R_1(s), R_2(s) \text{ are global equivalent}$$

$$\Updownarrow$$

$$R_1(s), R_2(s) \text{ are local equivalent at } \pi(s) \text{ for all prime } \pi(s) \in \mathbb{F}[s].$$

Example: $\mathbb{F} = \mathbb{R}$

$R(s) = \begin{bmatrix} \frac{s+1}{s} & -\frac{s}{s+1} \\ 1 & \frac{1}{(s+1)^2} \end{bmatrix} \in \mathbb{R}(s)^{2\times 2}$. The global Smith-McMillan form of $R(s)$ is

$$\begin{bmatrix} \frac{1}{s(s+1)^2} & 0 \\ 0 & (s+1)(s^2+1) \end{bmatrix}.$$

The local invariant rational functions of $R(s)$ at s are: $\frac{1}{s}, 1$.

The local invariant rational functions of $R(s)$ at $s+1$ are: $\frac{1}{(s+1)^2}, s+1$.

The local invariant rational functions of $R(s)$ at s^2+1 are: $1, s^2+1$.

The local invariant rational functions of $R(s)$ at any other prime polynomial are: $1, 1$.

Outline

1. Introduction
2. Global structure
3. Local structure
4. Infinite structure
5. Möbius transformations

$r(s) \in \mathbb{F}(s)$ has a zero (pole) at infinity $\Leftrightarrow r\left(\frac{1}{s}\right)$ has a zero (pole) at 0 (or at s).

Let $r(s) = \frac{p(s)}{q(s)} = \frac{p_n s^n + p_{n-1}s^{n-1} + \cdots + p_0}{q_d s^d + q_{d-1}s^{d-1} + \cdots + q_0}, \quad p_n, q_d \neq 0 \quad \Rightarrow$

$r\left(\frac{1}{s}\right) = \frac{p\left(\frac{1}{s}\right)}{q\left(\frac{1}{s}\right)} = \frac{p_n\left(\frac{1}{s}\right)^n + p_{n-1}\left(\frac{1}{s}\right)^{n-1} + \cdots + p_0}{q_d\left(\frac{1}{s}\right)^d + q_{d-1}\left(\frac{1}{s}\right)^{d-1} + \cdots + q_0} = \frac{p_n + p_{n-1}s + \cdots + p_0 s^n}{q_d + q_{d-1}s + \cdots + q_0 s^d} s^{d-n}.$

Recall that the local ring of $\mathbb{F}[s]$ at $\pi(s)$ is
$\mathbb{F}_\pi(s) = \{r(s) \in \mathbb{F}(s) : r(s)$ has no poles at $\pi(s)\}$.

Analogously, define "the local ring of $\mathbb{F}[s]$ at infinity":

$\mathbb{F}_\infty(s) = \{r(s) \in \mathbb{F}(s) : r(s)$ has no poles at infinity$\} =$
$\left\{r(s) \in \mathbb{F}(s) : r\left(\frac{1}{s}\right) \text{ has no poles at } s\right\} = \left\{r(s) \in \mathbb{F}(s) : r\left(\frac{1}{s}\right) \in \mathbb{F}_s(s)\right\} =$
$\left\{\frac{p(s)}{q(s)} \in \mathbb{F}(s) : \frac{p\left(\frac{1}{s}\right)}{q\left(\frac{1}{s}\right)} \in \mathbb{F}_s(s)\right\} = \left\{\frac{p(s)}{q(s)} \in \mathbb{F}(s) : d(q(s)) \geq d(p(s))\right\} = \mathbb{F}_{pr}(s)$

$$\mathbb{F}_{pr}(s) = \left\{ \frac{p(s)}{q(s)} \in \mathbb{F}(s) : d(q(s)) \geq d(p(s)) \right\}$$

⋆ Units: biproper rational functions=numerator and denominator have the same degree.

⋆ $\frac{p(s)}{q(s)} = \frac{p(s)}{q(s)} \frac{s^{d(q(s))-d(p(s))}}{s^{d(q(s))-d(p(s))}} = u_{pr}(s)\frac{1}{s^d}$, with $u_{pr}(s) = \frac{p(s)}{q(s)} s^{d(q(s))-d(p(s))}$, $d = d(q(s)) - d(p(s)) \geq 0$.

$$\mathbb{F}_{pr}(s) = \left\{ u_{pr}(s)\frac{1}{s^d} : u_{pr}(s) \text{ a unit}, d \in \mathbb{N} \cup \{0\} \right\} \cup \{0\}.$$

⋆ $\frac{u_{pr1}(s)}{s^{d_1}} \mid \frac{u_{pr2}(s)}{s^{d_2}} \Leftrightarrow d_1 \leq d_2 \left(\frac{u_{pr2}(s)}{s^{d_2}} = \frac{u_{pr2}(s)u_{pr1}(s)^{-1}}{s^{d_2-d_1}} \frac{u_{pr1}(s)}{s^{d_1}} \right)$.

⋆ $\mathbb{F}_{pr}(s)$ is a PID and its field of fractions is $\mathbb{F}(s)$: $\frac{\frac{u_{pr1}(s)}{s^{d_1}}}{\frac{u_{pr2}(s)}{s^{d_2}}} \in \mathbb{F}(s)$;

$$\frac{p(s)}{q(s)} = \frac{p(s)}{q(s)} \frac{s^{d(q(s))-d(p(s))}}{s^{d(q(s))-d(p(s))}} = \frac{u_{pr}(s)}{s^{d(q(s))-d(p(s))}} = \begin{cases} \frac{\frac{u_{pr}(s)}{s^{d(q(s))-d(p(s))}}}{1} & d(q(s)) - d(p(s)) \geq 0 \\ \frac{u_{pr}(s)}{\frac{1}{s^{d(p(s))-d(q(s))}}} & d(q(s)) - d(p(s)) \leq 0 \end{cases}$$

⋆ $\mathrm{Gl}_m(\mathbb{F}_{pr}(s))$: set of invertible matrices in $\mathbb{F}_{pr}(s)^{m\times m}$ (biproper matrices).

$$U(s) \in \mathrm{Gl}_m(\mathbb{F}_{pr}(s)) \Leftrightarrow U(s) \in \mathbb{F}_{pr}(s)^{m\times m} \text{ and } \det U(s) = u_{pr}(s).$$

$$\mathbb{F}_{pr}(s) = \left\{ \frac{p(s)}{q(s)} \in \mathbb{F}(s) : d(q(s)) \geq d(p(s)) \right\} = \left\{ \frac{u_{pr}(s)}{s^d} : u_{pr}(s) \text{ a unit}, d \geq 0 \right\} \cup \{0\}$$

Definition

$R_1(s), R_2(s) \in \mathbb{F}(s)^{m\times n}$ are **equivalent at infinity** *or* **equivalent in** $\mathbb{F}_{pr}(s)$ *if there exist $U(s) \in \mathrm{Gl}_m(\mathbb{F}_{pr}(s))$, $V(s) \in \mathrm{Gl}_n(\mathbb{F}_{pr}(s))$ so that*

$$R_2(s) = U(s)R_1(s)V(s).$$

Theorem (Smith-McMillan form at infinity)

Every $R(s) \in \mathbb{F}(s)^{m\times n}$ is equivalent at infinity to a matrix of the form

$$\begin{bmatrix} \mathrm{Diag}\left(\frac{1}{\frac{1}{s^{q_1}}}, \dots, \frac{1}{\frac{1}{s^{q_k}}}, 1, \dots, 1, \frac{\frac{1}{s^{-q_u}}}{1}, \dots, \frac{\frac{1}{s^{-q_r}}}{1}\right) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{Diag}(s^{q_1}, \dots, s^{q_r}) & 0 \\ 0 & 0 \end{bmatrix}$$

where $r = \mathrm{rank}\, R(s)$ and
$q_1 \geq \cdots \geq q_k > 0 = q_{k+1} = \cdots = q_{u-1} > q_u \geq \cdots \geq q_r$ are integers.

$$\mathbb{F}_{pr}(s) = \left\{ \frac{p(s)}{q(s)} \in \mathbb{F}(s) : d(q(s)) \geq d(p(s)) \right\} = \left\{ \frac{u_{pr}(s)}{s^d} : u_{pr}(s) \text{ a unit}, d \geq 0 \right\} \cup \{0\}$$

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Definition

$s^{q_1}, \dots, s^{q_r}$ are the **invariant rational functions at infinity** *of $R(s)$.*
$q_1, \dots, q_r$ are the **invariant orders at infinity** *of $R(s)$.*

Definition

$R(s)$ has $r - u + 1$ **zeros at infinity** *each one of order $-q_u, \dots, -q_r$.*
$R(s)$ has k **poles at infinity** *each one of order $q_1, \dots, q_k$.*

Remark

$R(s) \in \mathbb{F}_{pr}(s)^{m\times n} \Leftrightarrow q_1, \dots, q_r$ are nonpositive integers $\Leftrightarrow R(s)$ has no poles at infinity.

Infinite structure of polynomial matrices:

Let $P(s) = (p_{ij}(s))_{i,j} \in \mathbb{F}[s]^{m\times n}$. It turns out that $q_1 = d(P(s))$:

- $q_1 \geq 0$.
- Recall that the denominator of the first invariant rational function at infinity, $\frac{1}{s^{q_1}}$, is the monic least common denominator in $\mathbb{F}_{pr}(s)$ of $P(s)$.
- $p_{ij}(s) = \frac{1}{\frac{1}{p_{ij}(s)}} = \frac{1}{\frac{s^{d(p_{ij}(s))}}{p_{ij}(s)} \frac{1}{s^{d(p_{ij}(s))}}} = \frac{1}{\frac{u_{prij}(s)}{s^{d(p_{ij}(s))}}}$.
- Therefore, the monic least common denominator in $\mathbb{F}_{pr}(s)$ of $P(s)$ is $\frac{1}{s^{d(P(s))}}$.
- Thus, $\frac{1}{s^{q_1}} = \frac{1}{s^{d(P(s))}} \Rightarrow q_1 = d(P(s))$.

In consequence,

Nonconstant matrix polynomials always have poles at infinity and may also have zeros at infinity.

Let $P(s) \in \mathbb{F}[s]^{m\times m}$ and $P(s) = U(s)\,\mathrm{Diag}(s^{q_1}, \ldots, s^{q_m})V(s)$, $U(s), V(s) \in \mathrm{Gl}_m(\mathbb{F}_{pr}(s))$ $\Rightarrow \det P(s) = u_{pr}(s)s^{q_1+\cdots+q_m}$ and $d(\det P(s)) = q_1 + \cdots + q_m$.

Example: $P(s) = \begin{bmatrix} s & s \\ s^2 & s^2+1 \end{bmatrix}$. Its Smith-McMillan form at infinity is $\begin{bmatrix} s^2 & 0 \\ 0 & \frac{1}{s} \end{bmatrix}$.

Thus, $q_1 = 2, q_2 = -1$. It has one zero at infinity of order 1 and one pole at infinity of order 2.

- $\mathbb{F}_\pi(s) = \left\{u_\pi(s)\pi(s)^d : u_\pi(s) \text{ a unit}, d \geq 0\right\} \cup \{0\}$.

Theorem (Local Smith-McMillan form)

$$\begin{bmatrix} \mathrm{Diag}(\pi(s)^{h_1}, \ldots, \pi(s)^{h_r}) & 0 \\ 0 & 0 \end{bmatrix}$$

with $h_1 \leq \cdots \leq h_r$ integers.

- $\mathbb{F}_{pr}(s) = \left\{u_{pr}(s)\left(\frac{1}{s}\right)^d : u_{pr}(s) \text{ a unit}, d \geq 0\right\} \cup \{0\}$

Theorem (Smith-McMillan form at infinity)

$$\begin{bmatrix} \mathrm{Diag}(s^{q_1}, \ldots, s^{q_r}) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{Diag}\left(\left(\frac{1}{s}\right)^{-q_1}, \ldots, \left(\frac{1}{s}\right)^{-q_r}\right) & 0 \\ 0 & 0 \end{bmatrix}$$

with $q_1 \geq \cdots \geq q_r$ integers.

Remark: $\mathbb{F}_\pi(s)$ and $\mathbb{F}_{pr}(s)$ are analogous in the sense that:

- They are both local rings.
- $\pi(s)$ is to $\mathbb{F}_\pi(s)$ as $\frac{1}{s}$ is to $\mathbb{F}_{pr}(s)$.
- h_i and $-q_i$ play the same role in the Smith-McMillan forms.

Outline

$$\phi(s) = \frac{\alpha s + \beta}{\gamma s + \delta}, \quad \alpha\delta - \beta\gamma \neq 0 \quad \text{Möbius transformation}$$

$$\begin{array}{ccc} \mathbb{F}(s) & \to & \mathbb{F}(s) \\ r(s) & \to & r\left(\frac{\alpha s+\beta}{\gamma s+\delta}\right) \end{array} \quad \text{homomorphism}$$

- <u>If $\gamma = 0$</u> $\Rightarrow$ $\alpha, \delta \neq 0$ and $\phi(s) = \frac{\alpha}{\delta} s + \frac{\beta}{\delta} = cs + k, c \neq 0.$
- <u>If $\gamma \neq 0$</u> $\Rightarrow$ $\phi(s) = \frac{\alpha s+\beta}{\gamma s+\delta} = \frac{\beta\gamma - \alpha\delta}{\gamma^2}\left(\frac{\alpha\gamma}{\beta\gamma-\alpha\delta} + \frac{1}{s+\frac{\delta}{\gamma}}\right) = c(a + \frac{1}{s-b})$

$$f(s) = a + \frac{1}{s-b}, \quad a, b \in \mathbb{F}$$

$$f(s) = a + \frac{1}{s-b}, \quad a, b \in \mathbb{F}$$

Let $p(s) = p_d(s-a)^d + p_{d-1}(s-a)^{d-1} + \cdots + p_0 \in \mathbb{F}[s]$, $p_d \neq 0$.

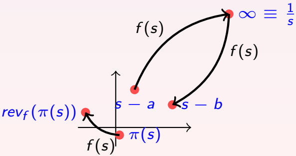
$p\left(f(s)\right) = p_d \frac{1}{(s-b)^d} + p_{d-1}\frac{1}{(s-b)^{d-1}} + \cdots + p_0 = \frac{p_d + p_{d-1}(s-b) + \cdots + p_0(s-b)^d}{(s-b)^d}$.

$rev_f(p(s)) = (s-b)^d p\left(f(s)\right) = p_0(s-b)^d + \cdots + p_{d-1}(s-b) + p_d$.

- $rev_f(p(s)) \in \mathbb{F}[s]$. Moreover, $d(rev_f(p(s)) \leq d(p(s))$.
- $d(rev_f(p(s)) = d(p(s)) \Leftrightarrow \gcd(p(s), s-a) = 1$.
- $\gcd(rev_f(p(s)), s-b) = 1$.
- If $p(s) \neq s-a$ is prime $\Rightarrow rev_f(p(s))$ is prime.
- If $p(s), q(s)$ are coprime $\Rightarrow rev_f(p(s))$, $rev_f(q(s))$ are coprime.

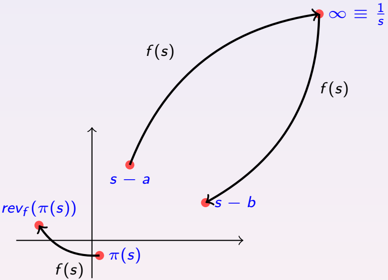
Let $\frac{p(s)}{q(s)}$. Then $\frac{p(f(s))}{q(f(s))} = \frac{rev_f(p(s))}{rev_f(q(s))}(s-b)^{d(q(s))-d(p(s))}$.

- If $\pi(s) \neq s-a$, $\frac{p(s)}{q(s)} \in \mathbb{F}_{\pi}(s) \Rightarrow \frac{p(f(s))}{q(f(s))} \in \mathbb{F}_{rev_f(\pi(s))}(s)$.
- If $\frac{p(s)}{q(s)} \in \mathbb{F}_{s-a}(s) \Rightarrow \frac{p(f(s))}{q(f(s))} \in \mathbb{F}_{pr}(s)$.
- If $\frac{p(s)}{q(s)} \in \mathbb{F}_{pr}(s) \Rightarrow \frac{p(f(s))}{q(f(s))} \in \mathbb{F}_{s-b}(s)$.



If $u(s)$ is a unit $\Rightarrow u(f(s))$ is a unit in the corresponding ring.

Let $f(s) = a + \frac{1}{s-b}$, $a, b \in \mathbb{F}$ and $rev_f(\pi(s)) = (s-b)^{d(\pi(s))}\pi\,(f(s))$.



Lemma

1. $\pi(s) \neq s - a$, *prime*. $U(s) \in \mathrm{Gl}_m(\mathbb{F}_\pi(s)) \Rightarrow U(f(s)) \in \mathrm{Gl}_m(\mathbb{F}_{rev_f(\pi(s))}(s))$.
2. $U(s) \in \mathrm{Gl}_m(\mathbb{F}_{s-a}(s)) \Rightarrow U(f(s)) \in \mathrm{Gl}_m(\mathbb{F}_{pr}(s))$.
3. $U(s) \in \mathrm{Gl}_m(\mathbb{F}_{pr}(s)) \Rightarrow U(f(s)) \in \mathrm{Gl}_m(\mathbb{F}_{s-b}(s))$.

Möbius transformations

$f(s) = a + \frac{1}{s-b}$, $a, b \in \mathbb{F}$, $\tilde{\pi}(s) = \frac{1}{\pi_0} rev_f(\pi(s)) = \frac{1}{\pi_0}(s-b)^{d(\pi(s))}\pi(f(s))$. $R(s) \in \mathbb{F}(s)^{m\times n}$.

Proposition

1. *Let $\pi(s) \neq s-a$, prime. If $\pi(s)^{h_1}, \ldots, \pi(s)^{h_r}$ are the invariant rational functions at $\pi(s)$ of $R(s) \Rightarrow \tilde{\pi}(s)^{h_1}, \ldots, \tilde{\pi}(s)^{h_r}$ are the invariant rational functions at $\tilde{\pi}(s)$ of $R(f(s))$.*
2. *If $(s-a)^{h_1}, \ldots, (s-a)^{h_r}$ are the invariant rational functions at $s-a$ of $R(s) \Rightarrow$ $s^{-h_1}, \ldots, s^{-h_r}$ are the invariant rational functions at infinity of $R(f(s))$.*
3. *If $s^{q_1}, \ldots, s^{q_r}$ are the invariant rational functions at infinity of $R(s) \Rightarrow$ $(s-b)^{-q_1}, \ldots, (s-b)^{-q_r}$ are the invariant rational functions at $s-b$ of $R(f(s))$.*

Proof.-

2. $(s-a)^{h_1}, \ldots, (s-a)^{h_r}$ are the invariant rational functions at $s-a$ of $R(s) \Rightarrow$

there exist $U(s) \in \mathrm{Gl}_m(\mathbb{F}_{s-a}(s))$, $V(s) \in \mathrm{Gl}_n(\mathbb{F}_{s-a}(s))$ such that

$$R(s) = U(s)\begin{bmatrix} \mathrm{Diag}((s-a)^{h_1}, \ldots, (s-a)^{h_r}) & 0 \\ 0 & 0 \end{bmatrix} V(s) \Rightarrow$$

there exist $U(f(s)) \in \mathrm{Gl}_m(\mathbb{F}_{pr}(s))$, $V(f(s)) \in \mathrm{Gl}_n(\mathbb{F}_{pr}(s))$ such that

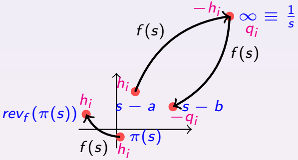
$$R(f(s)) = U(f(s))\begin{bmatrix} \mathrm{Diag}\left((s-b)^{-h_1}, \ldots, (s-b)^{-h_r}\right) & 0 \\ 0 & 0 \end{bmatrix} V(f(s)) =$$

$$U(f(s))\begin{bmatrix} \mathrm{Diag}\left((\frac{s-b}{s})^{-h_1}, \ldots, (\frac{s-b}{s})^{-h_r}\right) & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \mathrm{Diag}\left(s^{-h_1}, \ldots, s^{-h_r}\right) & 0 \\ 0 & 0 \end{bmatrix} V(f(s)) \Rightarrow$$

$s^{-h_1}, \ldots, s^{-h_r}$ are the invariant rational functions at infinity of $R(f(s))$.

Möbius transformations on the polynomial matrices:

$f(s) = a + \frac{1}{s-b}$, $a, b \in \mathbb{F}$, $\tilde{\pi}(s) = \frac{1}{\pi_0} rev_f(\pi(s)) = \frac{1}{\pi_0}(s-b)^{d(\pi(s))}\pi(f(s))$.
$P(s) \in \mathbb{F}[s]^{m\times n}$, $d = d(P(s))$.



Proposition

1. *Let $\pi(s) \neq s-a$, prime.*
   If $\pi(s)^{h_1}, \ldots, \pi(s)^{h_r}$ are the invariant rational functions at $\pi(s)$ of $P(s) \Rightarrow$ $\tilde{\pi}(s)^{h_1}, \ldots, \tilde{\pi}(s)^{h_r}$ are the invariant rational functions at $\tilde{\pi}(s)$ of $(s-b)^d P(f(s))$.
2. *If $(s-a)^{h_1}, \ldots, (s-a)^{h_r}$ are the invariant rational functions at $s-a$ of $P(s) \Rightarrow$ $s^{d-h_1}, \ldots, s^{d-h_r}$ are the invariant rational functions at infinity of $(s-b)^d P(f(s))$.*
3. *If $s^{q_1}, \ldots, s^{q_r}$ are the invariant rational functions at infinity of $P(s) \Rightarrow$ $(s-b)^{d-q_1}, \ldots, (s-b)^{d-q_r}$ are the invariant rational functions at $s-b$ of $(s-b)^d P(f(s))$.*

Definition

Let $P(s) \in \mathbb{F}[s]^{m\times n}$, $d = d(P(s))$. The **infinite elementary divisors** *of $P(s)$ are defined as the elementary divisors powers of s of the polynomial matrix $s^d P\left(\frac{1}{s}\right)$.*

Proposition

Let $P(s) \in \mathbb{F}[s]^{m\times n}$, $\operatorname{rank} P(s) = r$, $d(P(s)) = d$. Let $s^{e_1}, \ldots, s^{e_r}$ be its infinite elementary divisors (including exponents equal to zero) and $s^{q_1}, \ldots, s^{q_r}$ be its invariant rational functions at infinity. Then

$$e_i = d - q_i = q_1 - q_i, \quad i = 1, \ldots, r.$$

Proof.- $s^{e_1}, \ldots, s^{e_r}$ are the infinite elementary divisors (including exponents 0) of $P(s)$

$$\Updownarrow$$

$s^{e_1}, \ldots, s^{e_r}$ are the elementary divisors powers of s (including exponents 0) of $s^d P\left(\frac{1}{s}\right)$

$$\Updownarrow$$

$s^{e_1}, \ldots, s^{e_r}$ are the invariant rational functions at s of $s^d P\left(\frac{1}{s}\right)$

$$\Updownarrow$$

$s^{-e_1}, \ldots, s^{-e_r}$ are the invariant rational functions at infinity of $\frac{1}{s^d} P(s)$

$$\Updownarrow$$

$s^{d-e_1}, \ldots, s^{d-e_r}$ are the invariant rational functions at infinity of $P(s)$

$$\Updownarrow$$

$$q_i = d - e_i$$

OTHER STRUCTURES:

- **Structure in $\mathbb{F}_M(s)$**:
  $M = \{\pi(s) : \pi(s) \text{ some prime polynomials}\}$
  $\mathbb{F}_M(s) = \bigcap_{\pi(s)\in M} \mathbb{F}_\pi(s) = \left\{\frac{p(s)}{q(s)} \in \mathbb{F}(s) : \gcd(q(s), \pi(s)) = c \ \forall \pi(s) \in M\right\}$
  $R_1(s), R_2(s) \in \mathbb{F}(s)^{m\times n}$ are **equivalent in** $\mathbb{F}_M(s)$ if there exist $U(s) \in \mathrm{Gl}_m(\mathbb{F}_M(s))$ and $V(s) \in \mathrm{Gl}_n(\mathbb{F}_M(s))$ such that $R_2(s) = U(s)R_1(s)V(s)$.

- **Structure in $\mathbb{F}_M(s) \cap \mathbb{F}_{pr}(s)$**:
  $\mathbb{F}_M(s) \cap \mathbb{F}_{pr}(s) = \left(\bigcap_{\pi(s)\in M} \mathbb{F}_\pi(s)\right) \cap \mathbb{F}_{pr}(s) =$
  $\left\{\frac{p(s)}{q(s)} \in \mathbb{F}(s) : \gcd(q(s), \pi(s)) = c \ \forall \pi(s) \in M, d(q(s)) \geq d(p(s))\right\}$
  $R_1(s), R_2(s) \in \mathbb{F}(s)^{m\times n}$ are **equivalent in** $\mathbb{F}_M(s) \cap \mathbb{F}_{pr}(s)$ if there exist $U(s) \in \mathrm{Gl}_m(\mathbb{F}_M(s) \cap \mathbb{F}_{pr}(s))$ and $V(s) \in \mathrm{Gl}_n(\mathbb{F}_M(s) \cap \mathbb{F}_{pr}(s))$ such that $R_2(s) = U(s)R_1(s)V(s)$.

- **Wiener-Hopf structure**:
  $R_1(s), R_2(s) \in \mathbb{F}(s)^{m\times n}$ are **Wiener-Hopf equivalent with respect to** M if there exist $U(s) \in \mathrm{Gl}_m(\mathbb{F}_{M^c}(s) \cap \mathbb{F}_{pr}(s))$ and $V(s) \in \mathrm{Gl}_n(\mathbb{F}_M(s))$ such that $R_2(s) = U(s)R_1(s)V(s)$.

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