Second order pseudospectra of normal matrices

G. Armentia, J.M. Gracia, F.E. Velasco

Public University of Navarre, The University of the Basque Country, Spain

16th ILAS Conference, Pisa, June 24, 2010.

Contents

Pseudospectra

$A \in \mathbb{C}^{n\times n}, \varepsilon \geq 0$,

$\Lambda(A) = \{\lambda_1, \ldots, \lambda_v\}$ spectrum of $A, v \leq n$.

$\Lambda_2(A) := \{\lambda_0 \in \Lambda(A) \colon \mathrm{m}(\lambda_0, A) \geq 2\}$ 2nd order spectrum of A.

1st order ε-pseudospectrum:

$$\Lambda_{\varepsilon,1}(A) := \bigcup_{\substack{\Delta\in\mathbb{C}^{n\times n}\\ \|\Delta\|\leq\varepsilon}} \Lambda(A+\Delta)$$

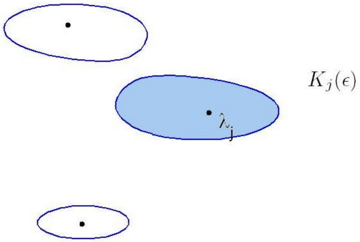
$\Lambda_\varepsilon(A) := \Lambda_{\varepsilon,1}(A)$, ordinary ε-pseudospectrum.

2nd order ε-pseudospectrum:

$$\Lambda_{\varepsilon,2}(A) := \bigcup_{\substack{\Delta\in\mathbb{C}^{n\times n}\\ \|\Delta\|\leq\varepsilon}} \Lambda_2(A+\Delta)$$

Definitions

Connected component $\mathcal{K}_j(\varepsilon)$ of the ε-pseudospectrum containing $\lambda_j \in \Lambda(A)$:



Diameter of $\mathcal{K}_j(\varepsilon)$:
$\delta_j(\varepsilon) := \sup\{|z-w| : z, w \in \mathcal{K}_j(\varepsilon)\}$

Closed disk:
$\mathcal{D}(\lambda_j, \varepsilon) := \{z \in \mathbb{C} : |z-\lambda_j| \leq \varepsilon\}$

Condition number of order $1/\nu_j$:
Let $\lambda_j \in \Lambda(A)$ s.t. $m_j := \mathrm{m}(\lambda_j, A)$

$$sv_{(A,\lambda_j)}(A') := \min\{\varepsilon \geq 0 \mid \exists\, \lambda'_1, \ldots, \lambda'_k \in \Lambda(A') \cap \mathcal{D}(\lambda_j, \varepsilon) : \textstyle\sum_{i=1}^{k} \mathrm{m}(\lambda'_i, A') \geq m_j\}$$

$$c_{\frac{1}{\nu_j}}(\lambda_j) := \lim_{\varepsilon \to 0^+} \max_{\substack{\Delta \in \mathbb{C}^{n\times n} \\ 0 < \|\Delta\| \leq \varepsilon}} \frac{sv_{(A,\lambda_j)}(A+\Delta)}{\|\Delta\|^{\frac{1}{\nu_j}}}$$

Sublevel sets

Fix $z \in \mathbb{C}$, then

$$\min_{z\in\Lambda(X)} \|X - A\| = \sigma_n(zI_n - A) =: h_1(z)$$

$$\Downarrow$$

$$\Lambda_\varepsilon(A) = \Lambda_{\varepsilon,1}(A) = \{z \in \mathbb{C} \colon h_1(z) \leq \varepsilon\}.$$

Theorem 1 (Malyshev, 1999)

Fix $z \in \mathbb{C}$, then

$$\min_{z\in\Lambda_2(X)} \|X - A\| = h_2(z),$$

where

$$h_2(z) := \max_{t\geq 0} \sigma_{2n-1} \begin{pmatrix} zI_n - A & tI_n \\ O & zI_n - A \end{pmatrix}.$$

$$\Downarrow$$

$$\Lambda_{\varepsilon,2}(A) = \{z \in \mathbb{C} \colon h_2(z) \leq \varepsilon\}.$$

Ordinary pseudospectra of normal matrices

Definition 2

$A \in \mathbb{C}^{n\times n}$ normal if $A^*A = AA^*$.

Theorem 3 (Pseudospectra of a normal matrix)

Let $A \in \mathbb{C}^{n\times n}$, $\Lambda(A) = \{\lambda_1, \ldots, \lambda_v\}$. *Then*

$$A \text{ normal} \quad \Longleftrightarrow \quad \forall \varepsilon \geq 0, \quad \Lambda_\varepsilon(A) = \bigcup_{j=1}^{v} \mathcal{D}(\lambda_j, \varepsilon).$$

Let $(\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$. $\forall \varepsilon \geq 0$,

$$\Lambda_\varepsilon\big(\operatorname{diag}(\mu_1, \ldots, \mu_n)\big) = \bigcup_{j=1}^{n} \mathcal{D}(\mu_j, \varepsilon).$$

$\boxed{\Longrightarrow}$ $\quad U^*AU$ diagonal $\quad \Rightarrow \quad \Lambda_\varepsilon(A) = \bigcup_{j=1}^{v} \mathcal{D}(\lambda_j, \varepsilon)$

Theorem 3 Proof, 1

Jordan decomposition.

$$A = \sum_{j=1}^{v} (\lambda_j P_j + N_j).$$

Theorem 4 (Condition number. Karow, 2003)

$$c_{\frac{1}{\nu_j}}(\lambda_j) = \begin{cases} \|P_j\|, & \text{if } \nu_j = 1. \\ \|N_j^{\nu_j - 1}\|^{\frac{1}{\nu_j}}, & \text{if } \nu_j \geq 2. \end{cases}$$

Theorem 5 (Normal matrices caracterization)

$$A \text{ normal} \quad \Leftrightarrow \quad A = \sum_{j=1}^{v} \lambda_j P_j, \text{ with } P_i P_j = \delta_{ij} P_j \text{ and } P_j^* = P_j.$$

Theorem 3 Proof, 2

Theorem 6 (Diameter derivative)

Let $\lambda_j \in \Lambda(A)$,

$$\delta_j'(0^+) = \begin{cases} 2\ c_1(\lambda_j), & \textit{if } \nu_j = 1. \\ \infty, & \textit{if } \nu_j \geq 2. \end{cases}$$

Lemma 7 (Orthogonal proyector)

Let $P \in \mathbb{C}^{n\times n}$, *s. t.* $P^2 = P$ *and* $\|P\| = 1$, *then* $P^* = P$.

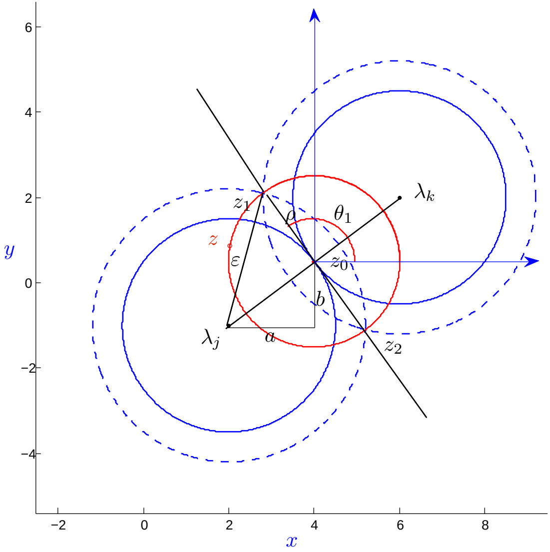
$\boxed{\Longleftarrow}$ (Theorem 3 sufficient condition proof sketch)

$$\text{If } \Lambda_\varepsilon(A) = \bigcup_{j=1}^{v} \mathcal{D}(\lambda_j, \varepsilon) \Rightarrow \delta_j'(\varepsilon) = 2 \Rightarrow \delta_j'(0^+) = 2 \Rightarrow \boxed{\nu_j = 1}$$

$$\Downarrow$$

$$2\ c_1(\lambda_j) = 2 \Rightarrow \boxed{c_1(\lambda_j) = 1} \Rightarrow \|P_j\| = 1 \overset{\text{Lemma 7}}{\Rightarrow} P_j^* = P_j \Rightarrow A \text{ normal.} \quad \square$$

Circles



6
4
2
0
−2
−4
−2
0
2
4
6
8
x
y
λ_k
λ_j
z_1
z_2
z_0
z
ρ
θ_1
ε
a
b

Point where the maximum is attained

$$h_2(z) := \max_{t \geq 0} \sigma_{2n-1} \begin{pmatrix} zI_n - A & tI_n \\ O & zI_n - A \end{pmatrix}$$

$$t \mapsto \sigma_{2n-1} \begin{pmatrix} zI_n - A & tI_n \\ O & zI_n - A \end{pmatrix}, \quad \text{unimodal function}$$

$$t_0 := \operatorname*{argmax}_{t \geq 0} \sigma_{2n-1} \begin{pmatrix} zI_n - A & tI_n \\ O & zI_n - A \end{pmatrix}$$

Fix $z \in \mathbb{C}$. Let $\lambda_j \in \Lambda(A)$ s.t.

$$|z - \lambda_j| \leq |z - \lambda_i|, \quad \forall i \in \Lambda(A) \setminus \{\lambda_j\}$$

Case 1. λ_j multiple eigenvalue, $\Rightarrow \quad t_0 = 0, \quad h_2(z) = \sigma_n(zI_n - A) = h_1(z)$.
Case 2. λ_j simple eigenvalue, let $\lambda_k \in \Lambda(A)$ be s.t.

$$|z - \lambda_j| \leq |z - \lambda_k| \leq |z - \lambda_i| \quad \forall \lambda_i \in \Lambda(A) \setminus \{\lambda_j, \lambda_k\}, \quad \Rightarrow$$

$$h_2(z) = \sigma_{2n-1} \begin{pmatrix} zI_n - A & t_0 I_n \\ O & zI_n - A \end{pmatrix}, \text{ with } t_0 = \boxed{\frac{\left| |z - \lambda_k|^2 - |z - \lambda_j|^2 \right|}{\sqrt{2\left(|z - \lambda_k|^2 + |z - \lambda_j|^2\right)}}}$$

Invariance over circles

$$f(\theta) := h_2^2\left(z_0 + \rho e^{i\theta}\right), \qquad 0 \le \theta \le 2\pi,$$

where

$z_0 := \frac{\lambda_j + \lambda_k}{2}$.

Lemma 8

The function $f : [0, 2\pi] \to \mathbb{R}$ *is constant. Specifically,* $f(\theta) \equiv \varepsilon^2$.

Observation

This implies the function $h_2 : \mathbb{C} \to \mathbb{R}$ is constant over the circle $C_{z_0, \rho(\varepsilon)}$ of centre z_0 and radius $\rho(\varepsilon)$ $\Longrightarrow$

$$C_{z_0, \rho(\varepsilon)} \subseteq \{z \in \mathbb{C} : h_2(z) = \varepsilon\},$$

$$\rho(\varepsilon)^2 + \left(\frac{|\lambda_k - \lambda_j|}{2}\right)^2 = \varepsilon^2, \quad \text{Pythagoras' theorem.}$$

Main results. $\#\big(\Lambda(A)\big) = 5$, $\#(\mathcal{S}) = 10$

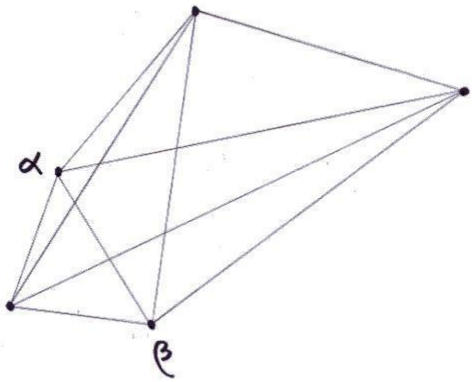


Figure: Line segments between eigenvalues.

Notations and definitions

$A \in \mathbb{C}^{n\times n}$ normal, $v := \#\big(\Lambda(A)\big), v \leq n$,

$$\mathcal{S} := \{[\alpha, \beta] \colon \alpha, \beta \in \Lambda(A), \alpha \neq \beta\}$$

set of line segments

$$[\alpha, \beta] := \{(1-t)\alpha + t\beta \colon 0 \leq t \leq 1\}$$

So,

$$\#(\mathcal{S}) = \binom{v}{2} =: j_\ell$$

Let us observe that $[\alpha, \beta] = [\beta, \alpha]$

For each $[\alpha, \beta] \in \mathcal{S}$ we agree that $\alpha \overset{\text{lex}}{\leq} \beta, \alpha \neq \beta$.

Ordered list of the elements of $\mathcal{S}$

$$\overbrace{[\alpha_1,\beta_1],\ldots,[\alpha_{j_1},\beta_{j_1}]}^{j_1},\overbrace{[\alpha_{j_1+1},\beta_{j_1+1}],\ldots,[\alpha_{j_2},\beta_{j_2}]}^{j_2-j_1},\ldots$$

$$\ldots,[\alpha_{j_{k-1}},\beta_{j_{k-1}}],\overbrace{[\alpha_{j_{k-1}+1},\beta_{j_{k-1}+1}],\ldots,[\alpha_{j_k},\beta_{j_k}]}^{j_k-j_{k-1}},\ldots$$

$$\ldots,[\alpha_{j_{\ell-1}},\beta_{j_{\ell-1}}],\overbrace{[\alpha_{j_{\ell-1}+1},\beta_{j_{\ell-1}+1}],\ldots,[\alpha_{j_\ell},\beta_{j_\ell}]}^{j_\ell-j_{\ell-1}}.$$

$$\overbrace{|\beta_1-\alpha_1|=\cdots=|\beta_{j_1}-\alpha_{j_1}|}^{j_1}<\overbrace{|\beta_{j_1+1}-\alpha_{j_1+1}|=\cdots=|\beta_{j_2}-\alpha_{j_2}|}^{j_2-j_1}<\cdots$$

$$\cdots=|\beta_{j_{k-1}}-\alpha_{j_{k-1}}|<\overbrace{|\beta_{j_{k-1}+1}-\alpha_{j_{k-1}+1}|=\cdots=|\beta_{j_k}-\alpha_{j_k}|}^{j_k-j_{k-1}}<\cdots$$

$$\cdots=|\beta_{j_{\ell-1}}-\alpha_{j_{\ell-1}}|<\overbrace{|\beta_{j_{\ell-1}+1}-\alpha_{j_{\ell-1}+1}|=\cdots=|\beta_{j_\ell}-\alpha_{j_\ell}|}^{j_\ell-j_{\ell-1}}.$$

Step function $p\colon [0,\infty) \to \mathbb{N}$

$$p(\varepsilon) := \begin{cases} 0, & \text{if} \quad 0 \le \varepsilon < \dfrac{|\beta_{j_1} - \alpha_{j_1}|}{2}. \\ r-1, & \text{if} \quad \dfrac{|\beta_{j_{r-1}} - \alpha_{j_{r-1}}|}{2} \le \varepsilon < \dfrac{|\beta_{j_r} - \alpha_{j_r}|}{2}, \quad r = 2, \dots, \ell. \\ \ell, & \text{if} \quad \dfrac{|\beta_{j_\ell} - \alpha_{j_\ell}|}{2} \le \varepsilon. \end{cases}$$

Theorem 9 (Main result)

$\forall \varepsilon \ge 0$,

$$\Lambda_{\varepsilon,2}(A) = \bigcup_{k=1}^{p(\varepsilon)} \left[\bigcup_{i=j_{k-1}+1}^{j_k} \mathcal{D}\left(\frac{\alpha_i + \beta_i}{2}, \rho_k(\varepsilon) \right) \right] \cup \left[\bigcup_{\mu \in \Lambda_2(A)} \mathcal{D}(\mu, \varepsilon) \right],$$

where

$$\rho_k(\varepsilon) := \sqrt{\varepsilon^2 - \frac{|\beta_{j_k} - \alpha_{j_k}|^2}{4}}, \quad k = 1, \dots, \ell,$$

$$\bigcup_{k=1}^{0} S_k := \emptyset, \quad j_0 := 0.$$

Multiple eigenvalues

Corollary 10 (Simple eigenvalues)

$\forall \varepsilon \geq 0$,

$$\Lambda_{\varepsilon,2}(A) = \bigcup_{k=1}^{p(\varepsilon)} \left[\bigcup_{i=j_{k-1}+1}^{j_k} \mathcal{D}\left(\frac{\alpha_i + \beta_i}{2}, \rho_k(\varepsilon) \right) \right].$$

Corollary 11 (All eigenvalues multiple)

If $\Lambda_2(A) = \Lambda(A) = \{\lambda_1, \ldots, \lambda_v\}$,
$\forall \varepsilon \geq 0$,

$$\Lambda_{\varepsilon,2}(A) = \bigcup_{i=1}^{v} \mathcal{D}(\lambda_i, \varepsilon).$$

Explanation of the main result

$\forall \varepsilon \geq 0,$

$\Lambda_{\varepsilon,2}(A) =$

$$\begin{cases} \emptyset, & \text{if } 0 \leq \varepsilon < \frac{\left|\beta_{j_1} - \alpha_{j_1}\right|}{2}. \\ \left[\bigcup_{i=1}^{j_1} \mathcal{D}\left(\frac{\alpha_i+\beta_i}{2}, \rho_1(\varepsilon)\right)\right], & \text{if } \frac{\left|\beta_{j_1} - \alpha_{j_1}\right|}{2} \leq \varepsilon < \frac{\left|\beta_{j_2} - \alpha_{j_2}\right|}{2}. \\ \left[\bigcup_{i=1}^{j_1} \mathcal{D}\left(\frac{\alpha_i+\beta_i}{2}, \rho_1(\varepsilon)\right)\right] \cup \left[\bigcup_{i=j_1+1}^{j_2} \mathcal{D}\left(\frac{\alpha_i+\beta_i}{2}, \rho_2(\varepsilon)\right)\right], & \text{if } \frac{\left|\beta_{j_2} - \alpha_{j_2}\right|}{2} \leq \varepsilon < \frac{\left|\beta_{j_3} - \alpha_{j_3}\right|}{2}. \\ \vdots & \vdots \end{cases}$$

Bibliography

M. Karow
Geometry of spectral sets.
Ph. D. Thesis, Universität Bremen, 2003.

A.N. Malyshev
A formula for the 2-norm distance from a matrix to the set of matrices with multiple eigenvalues.
Numer. Math. (3) 83 (1999) 443–454.

The End

Thanks for your attention!