The Inverse Eigenvalue Problem for Real and Symmetric Quadratic Matrix Polynomials

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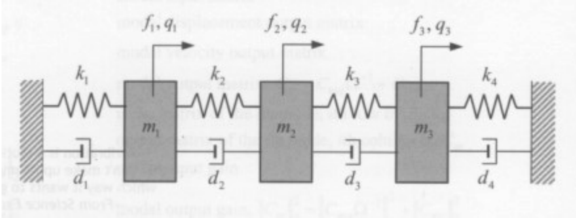
Outline

- Quadratic systems and the quadratic eigenvalue problem
- The inverse quadratic eigenvalue problem: admissible spectral data
- Eigenvalues and sign characteristic
- Selfadjoint Jordan triples
- An orthogonality property of eigenvectors
- A procedure to construct vibrating systems: semisimple case

Quadratic systems: Examples (mass-damper-spring)



$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = f(t)$$

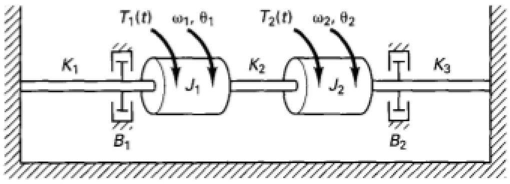
$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix},$$ **Mass matrix**

$$D = \begin{bmatrix} d_1+d_2 & -d_2 & 0 \\ -d_2 & d_2+d_3 & -d_3 \\ 0 & -d_3 & d_3+d_4 \end{bmatrix},$$ **Damping matrix**

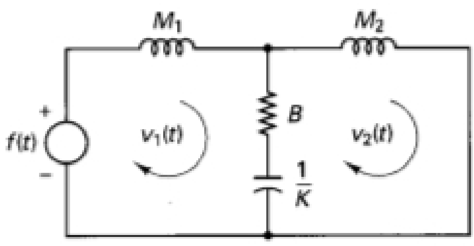
$$K = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix},$$ **Stiffness matrix**

$$f(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} f_3(t), \quad (f_1(t) = f_2(t) = 0)$$

Quadratic systmes: Examples (rotating rigid bodies and electric circuits)

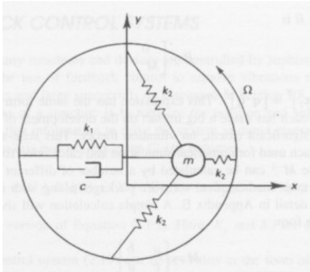


$$\begin{aligned} J_1\ddot{\theta}_1(t) + B_1\dot{\theta}_1(t) + K_1\theta_1(t) + K_2(\theta_1(t) - \theta_2(t)) &= T_1(t) \\ J_2\ddot{\theta}_2(t) + B_2\dot{\theta}_2(t) + K_2(\theta_2(t) - \theta_1(t)) + K_3\theta_2(t) &= T_2(t) \end{aligned}$$



$$\begin{aligned} M_1\ddot{Q}_1(t) + B(\dot{Q}_1(t) - \dot{Q}_2(t)) + K(D(t) - M(t)) &= f(t) \\ M_2\ddot{Q}_2(t) + B(\dot{Q}_2(t) - \dot{Q}_1(t)) + K(M(t) - D(t)) &= 0 \end{aligned}$$

Quadratic systems: Examples (simplified spinning disk)



$$M\ddot{q}(t) + (D+G)\dot{q}(t) + Kq(t) = 0$$

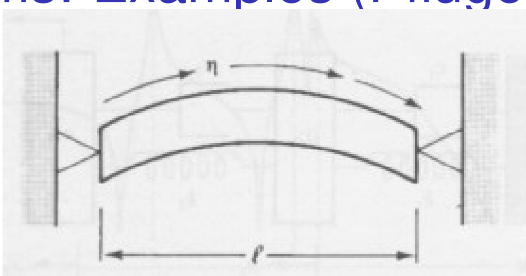
$$M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$$D = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}, \quad G = 2m\Omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 + k_2 - m\Omega^2 & 0 \\ 0 & 2k_2 - m\Omega^2 \end{bmatrix}$$

Ω=constant angular velocity, $q(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

Quadratic systems: Examples (Pflüger's rod)



$$M\ddot{q}(t) + D\dot{q}(t) + (K+H)q(t) = 0$$

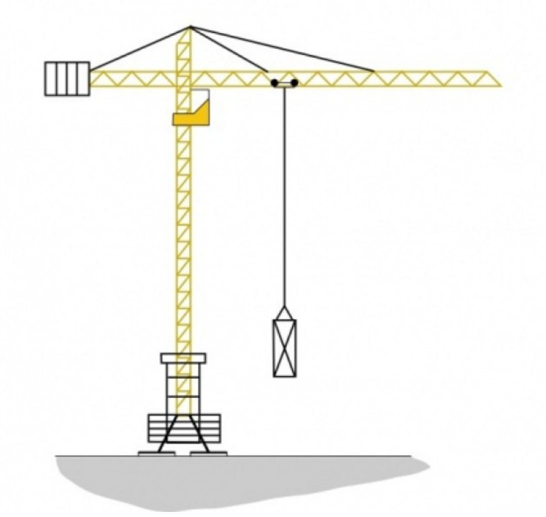
$$M = \begin{bmatrix} \frac{m}{2} & 0 \\ 0 & \frac{m}{2} \end{bmatrix}$$

$$D = 0$$

$$K = \begin{bmatrix} \frac{EI\pi^4}{2\ell^3} - \frac{pi}{4}\eta & -\frac{20}{9}\eta \\ -\frac{20}{9}\eta & \frac{8EI\pi^4}{\ell^3} - \eta\pi^2 \end{bmatrix}, \quad H = \frac{12\eta}{9}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

EI=flexural rigidity, m= mass density, $q(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$= displacements of two points of the rod, η=follower force.

Quadratic systems: Other examples



(a) Segway

(b) Saturn rocket

Quadratic systems

Euler-Lagrange equations applied to many physical systems (after discretization or linearization) lead to:

$$M\ddot{q}(t) + (D + G)\dot{q}(t) + (K + H)q(t) = f(t)$$

where

- M is symmetric and positive definite
- D and K are symmetric and positive (semi)definite
- G= gyroscopic matrix is skew-symmetric
- H=circulatory matrix is skew-symmetric

Vibrating systems : $G = H = 0$

Solving quadratic systems

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = 0 \qquad \det M \neq 0$$

$$\dot{y}(t) = \underbrace{\begin{bmatrix} 0 & I_n \\ -M^{-1}K & -M^{-1}D \end{bmatrix}}_{C_R} y(t), \quad y(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$$

- General solution of $\dot{y}(t) = C_R(t)y(t)$: if $T^{-1}C_RT = J$ (Jordan form):

$$y(t) = Te^{Jt}c, \quad c \in \mathbb{C}^{2n\times 1}$$

- General solution of $M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = 0$:

$$x(t) = Xe^{Jt}c, \quad c \in \mathbb{C}^{2n\times 1}, \quad T = \begin{bmatrix} X \\ \widetilde{X} \end{bmatrix}$$

$$\boxed{J = \text{Jordan form of } Q(\lambda)}$$

$Q(\lambda) = M\lambda^2 + D\lambda + K$

Jordan Pairs

- The transforming matrix $T = \begin{bmatrix} X \\ \widetilde{X} \end{bmatrix}$ is

$$\begin{bmatrix} 0 & I_n \\ -M^{-1}K & -M^{-1}D \end{bmatrix} \begin{bmatrix} X \\ \widetilde{X} \end{bmatrix} = \begin{bmatrix} X \\ \widetilde{X} \end{bmatrix} J \Rightarrow T = \begin{bmatrix} X \\ XJ \end{bmatrix}$$

and

$$MXJ^2 + DXJ + KX = 0$$

- X= matrix of Jordan chains of $Q(\lambda)$, $\operatorname{rank} X = n$.
- (X, J)= **Jordan pair** of $Q(\lambda)$:
  1. $\det \begin{bmatrix} X \\ XJ \end{bmatrix} \neq 0.$
  2. $MXJ^2 + DXJ + KX = 0$

  If $J = \operatorname{Diag}(\lambda_1, \ldots, \lambda_{2n})$ and $X(:, j) = x_j$:

$$MXJ^2 + DXJ + KX = 0 \Leftrightarrow (M\lambda_j^2 + D\lambda_j + K)x_j = 0$$

Quadratic Eigenvalue Problem (QEP)

Given a quadratic system, describe its behaviour

Given an $n \times n$ **real** quadratic matrix polynomial

$$Q(\lambda) = M\lambda^2 + D\lambda + K,$$

find complex numbers $\lambda_0 \in \mathbb{C}$ and non-zero vectors $x_0 \in \mathbb{C}^{n\times 1}$ such that

$$\det Q(\lambda_0) = 0 \quad \text{and} \quad Q(\lambda_0)x_0 = 0$$

- λ_0=eigenvalue of $Q(\lambda)$: $\lambda_0 \in \Lambda(Q)$
- x_0= eigenvector of $Q(\lambda)$ associated to λ_0

Inverse Real Quadratic Eigenvalue Problem (IRQEP)

Construct real quadratic systems with a desired behaviour

Design a procedure to construct real quadratic matrix polynomials with prescribed **admissible** spectral data: eigenvalues and (generalized) eigenvectors.

Key word: Admissible

- J is a Jordan matrix of some quadratic matrix polynomials $Q(\lambda)$ $(n \times n)$ if and only if the number of Jordan blocks for each eigenvalue is $\leq n$.
- Exampe: $J = \mathrm{Diag}(1, 1, 1-i, 1+i)$

$$X_c = \begin{bmatrix} -0.8045 & 0.8351 & 0.2157 - 1.1480i & -0.3680 - 1.2952i \\ 0.6966 & -0.2437 & -1.1658 + 0.1049i & 2.6986 - 0.0036i \end{bmatrix}$$

$$X_c = \begin{bmatrix} -0.8045 & 0.8351 & 0.2157 - 1.1480i & -0.3680 - 1.2952i \\ 0.6966 & -0.2437 & -1.1658 + 0.1049i & 1.3493 - 0.0018i \end{bmatrix}$$

$$\begin{bmatrix} -1.0616 & -0.6156 \\ 2.3505 & 0.7481 \end{bmatrix}\lambda^2 + \begin{bmatrix} 2.4642 & 0.3043 \\ -4.8156 & 0.7299 \end{bmatrix}\lambda + \begin{bmatrix} -1.4026 & 0.3113 \\ 2.4651 & -1.4780 \end{bmatrix}$$

The construction of a **real** quadratic system

1. Complex Jordan pair $\rightarrow$ Real Jordan pair

$$J_c = \mathrm{Diag}(1,1,1-i,1+i) \quad \rightarrow \quad J = \left(1, 1, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right)$$

$$X_c \quad \rightarrow \quad X = \begin{bmatrix} X_c(:,1) & X_c(:,2) & \sqrt{2}\mathrm{imag}(\mathrm{X_c}(:,4)) & \sqrt{2}\mathrm{real}(\mathrm{X_c}(:,4)) \end{bmatrix}$$

2. Take any invertible M
3. Recall $\begin{bmatrix} X \\ XJ \end{bmatrix}$ is invertible. $\begin{bmatrix} X \\ XJ \end{bmatrix} J = \begin{bmatrix} 0 & I \\ A & B \end{bmatrix} \begin{bmatrix} X \\ XJ \end{bmatrix}$
4. Define: $\begin{bmatrix} -M^{-1}K & -M^{-1}D \end{bmatrix} = [A \quad B] = XJ^2 \begin{bmatrix} X \\ XJ \end{bmatrix}^{-1}$. Then

$$[K \quad D] = -MXJ^2 \begin{bmatrix} X \\ XJ \end{bmatrix}^{-1}$$

MATLAB

The Inverse Real and Symmetric Quadratic Eigenvalue Problem (IRSQEP)

Construct real and symmetric quadratic matrix polynomials (with definiteness constraints on the coefficients, possibly) with prescribed Jordan form.

Spectral data: J **real** Jordan matrix

- There is no quadratic polynomial with positive coefficients and roots $+1$ and -1: $\pm(\lambda^2 - 1)$.
- Admissible spectral data for vibrating systems? $\longrightarrow$ *direct problem*.

Assumption: $Q(\lambda)$ semisimple.

Antecedents

- Chu and Golub(2005): *Inverse Eigenvalue Problems*.
- Bai, Chu, Sun(2007), Chu,Kuo,Lin(2004), . . . : partial list of eigenvalues and eigenvectors.
- Cai, Xu, Kuo, Lin(2009), Yuan, Dai(2011): complete list of eigenvalues and eigenvectors in the simple case with no definiteness constrains.
- Ram,Elhay(1996), Bai(2008): Tridiagonal systems with partial list of eigenvalues and eigenvectors.
- Chu(2008): Inverse Quadratic Problem and Model Updating.
- Chu,Datta(1996), Datta, Elhay, Ram(1997), Nichols,Kautsky(2001), Chu(2002), Abdelaziz(2016): Pole placement problem.
- Lancaster, Prells, Maroulas: $2n$ eigenvalues are prescribed with partial solutions.

A Theorem of Rellich

Let $Q(\lambda) = M\lambda^2 + D\lambda + K$ with M, D, K $n \times n$ Hermitian. Then there exist ***real-valued analytic*** functions $\mu_1(\lambda)$,. . . , $\mu_n(\lambda)$, $\lambda \in \mathbb{R}$, and an $n \times n$ ***complex-valued analytic*** matrix functions $U(\lambda)$ such that $U(\lambda)U(\lambda)^* = I_n$ and

$$Q(\lambda) = U(\lambda)^* \operatorname{Diag}(\mu_1(\lambda), \ldots, \mu_n(\lambda))U(\lambda), \quad \lambda \in \mathbb{R}$$

$\mu_1(\lambda), \ldots, \mu_n(\lambda)$=**eigenvalue functions** of $Q(\lambda)$

- $\det Q(\lambda_0) = 0$ if and only if $\mu_j(\lambda_0) = 0$ for some j
- $\dim \operatorname{Ker} Q(\lambda_0) = \#\{j : \mu_j(\lambda_0) = 0\}$
- $\det(\mu_j(\lambda)I_n - Q(\lambda)) = 0$, $\lambda \in \mathbb{R}$, for all j

Semisimple case: For each **real** $\lambda_k \in \Lambda(Q)$ there is j such that

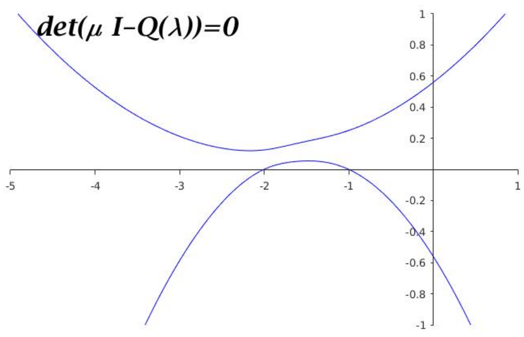
$$\mu_j(\lambda) = (\lambda - \lambda_k)\nu_j(\lambda), \quad \nu_j(\lambda_k) \neq 0$$
$$\epsilon(\lambda_k) = \text{sgn}(\nu_j(\lambda_k)) = \text{sign characteristic of } Q(\lambda) \text{ at } \lambda_k$$
$$\nu_j(\lambda_k) = \mu_j'(\lambda_k)$$

MATLAB

Eigenvalue Functions: Example I

$$Q(\lambda) = \begin{bmatrix} -1/4 & 1/8 \\ 1/8 & 1/16 \end{bmatrix} \lambda^2 + \begin{bmatrix} -3/4 & 3/8 \\ 3/8 & 5/16 \end{bmatrix} \lambda + \begin{bmatrix} -1/2 & 1/4 \\ 1/4 & 1/2 \end{bmatrix}$$

$$J = \mathrm{Diag}\left(-2, -1, \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix} \right)$$



positive type

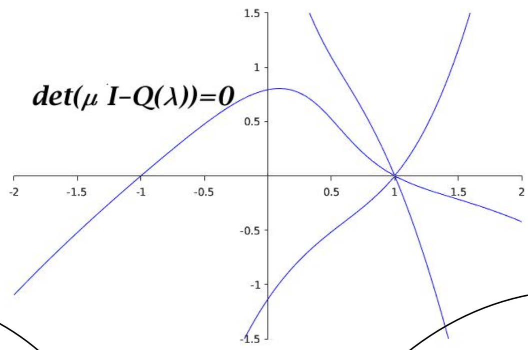
negative type

$$\epsilon(-2) = +1, \quad \epsilon(-1) = -1$$

Example II

$$Q(\lambda) = \begin{bmatrix} -0.5 & 2 & 0.5 \\ 2 & 1.4 & -0.5 \\ 0.5 & -0.5 & -0.5 \end{bmatrix} \lambda^2 + \begin{bmatrix} -1.0 & -3.7 & 0 \\ -3.7 & -1.8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \lambda + \begin{bmatrix} 1.5 & 1.7 & -0.5 \\ 1.7 & 0.4 & 0.5 \\ -0.5 & 0.5 & 0.5 \end{bmatrix}$$

$$J = \mathrm{Diag}\left(-1, 1, 1, 1, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right)$$



positive type

negative type

$$\epsilon(-1) = +1, \epsilon(1) = +1, \epsilon(1) = -1, \epsilon(1) = -1$$

Sign Characteristic

Semisimple Matrix Polynomials

Quadratic Case

$$\epsilon_i = \pm 1 \longrightarrow \sum_{j=1}^{2q} \epsilon_j = 0 \Rightarrow \begin{cases} \epsilon_j = +1, 1 \le j \le q \\ \epsilon_j = -1, q+1 \le j \le 2q \end{cases}$$

$$\begin{array}{cccccccccccc} & & \epsilon_1 & \epsilon_2 & \dots & \epsilon_{2q} \leftarrow \text{Sign characteristic} \\ & & \uparrow & \uparrow & & \uparrow \\ \Lambda(Q) & = (& r_1, & r_2, & \dots, & r_{2q}, & \zeta_1, & \bar{\zeta}_1, & \dots, & \zeta_s, & \bar{\zeta}_s &), \\ & & & \in \mathbb{R} & & & \zeta_j = \alpha_j + i\beta_j, & \beta_j > 0 \end{array}$$

Asymptotic Behaviour of the Eigenvalue Functions

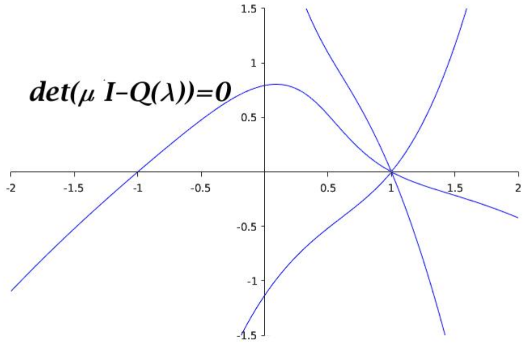
$L(\lambda) = L_\ell \lambda^\ell + L_{\ell-1}\lambda^{\ell-1} + \cdots + L_0$ symmetric with $\det L_\ell \neq 0$. $\mu_1(\lambda)$, ..., $\mu_n(\lambda)$ its eigenvalue functions and $\lambda_{\max}$ the highest real eigenvalue of $L(\lambda)$. If $(\pi, n-\pi, 0)$ is the inertia of L_ℓ then, for $\lambda > \lambda_{\max}$, π eigenvalue functions are positive ($\mu_j(\lambda) > 0$) and $n - \pi$ are negative ($\mu_j(\lambda) < 0$).

If $L_\ell > 0$ then for $\lambda > \lambda_{\max}$, $\mu_j(\lambda) > 0$, $j = 1, \dots, n$

Example II

$$Q(\lambda) = \begin{bmatrix} -0.5 & 2 & 0.5 \\ 2 & 1.4 & -0.5 \\ 0.5 & -0.5 & -0.5 \end{bmatrix} \lambda^2 + \begin{bmatrix} -1.0 & -3.7 & 0 \\ -3.7 & -1.8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \lambda + \begin{bmatrix} 1.5 & 1.7 & -0.5 \\ 1.7 & 0.4 & 0.5 \\ -0.5 & 0.5 & 0.5 \end{bmatrix}$$

$$J = \mathrm{Diag}\left(-1, 1, 1, 1, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right)$$



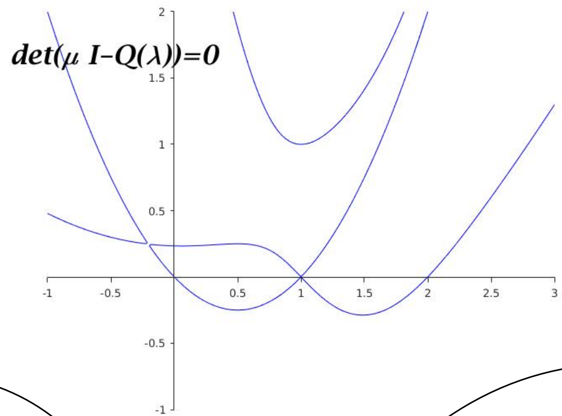
$$\epsilon(-1) = +1, \epsilon(1) = +1, \epsilon(1) = -1, \epsilon(1) = -1$$

$$\Lambda(L_2) = \{-2.0939, -0.2090, 2.7039\}, \Lambda(L_0) = \{-1.0984, 0.7969, 2.7027\}$$

Example III: Positive Mass Matrix

$$Q(\lambda) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & \frac{3}{2} \end{bmatrix} \lambda^2 + \begin{bmatrix} -3 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & -3 \end{bmatrix} \lambda + \begin{bmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & \frac{5}{2} \end{bmatrix}$$

$$J = \mathrm{Diag}\left(0, 1, 1, 2, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)$$



negative type

positive type

$$\epsilon(0) = -1, \epsilon(1) = -1, \epsilon(1) = +1, \epsilon(2) = +1$$

$$\Lambda(L_2) = \{1, 0.2192\ldots, 2.2807\ldots\}, \Lambda(L_0) = \{0, 0.2344, 4.2656\}$$

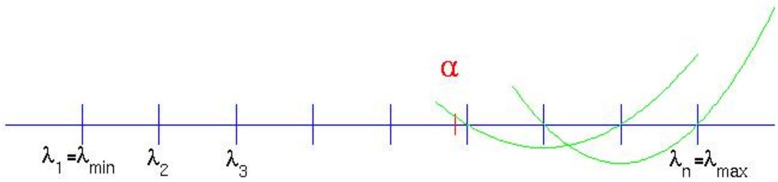
The Sign Characteristic when $L_\ell > 0$

$L(\lambda) = L_\ell\lambda^\ell + L_{\ell-1}\lambda^{\ell-1} + \cdots + L_0$ symmetric and **semisimple** with $L_\ell > 0$. $\lambda_{\min}$, $\lambda_{\max}$ minimal and maximal real eigenvalues of $L(\lambda)$. For any $\alpha \leq \lambda_{\max}$

- $p(\alpha) =$ number of real eigenvalues (counting multiplicities) of $L(\lambda)$ of positive type in $(\alpha, \lambda_{\max}]$
- $n(\alpha) =$ number of real eigenvalues (counting multiplicites) of $L(\lambda)$ of negative type in $[\alpha, \lambda_{\max}]$

Then

$$n(\alpha) \leq p(\alpha), \quad \forall\alpha \in [\lambda_{\min}, \lambda_{\max}]$$



The IRSQEP with Positive Definite Mass Matrix

There exists a semisimple real symmetric $Q(\lambda) = M\lambda^2 + D\lambda + K$ with $M > 0$ and Jordan matrix J if and only if the sign characteristic can be assigned such that

$$n(\alpha) \leq p(\alpha)$$

for all $\alpha \in [\lambda_{\min}, \lambda_{\max}]$. In this case, $Q(\lambda)$ can be constructed diagonal.

$n(\alpha) \leq p(\alpha) \Rightarrow$ real eigenvalues can be taken in pairs $(\lambda_{2i-1}, \lambda_{2i})$:

- $\max(\lambda_{2i-1}, \lambda_{2i})$ of positive type, and
- $\min(\lambda_{2i-1}, \lambda_{2i})$ of negative type

Consequences:

- $Q(\lambda) = \mathrm{Diag}\,\{(\lambda - \lambda_{2i-1})(\lambda - \lambda_{2i})\} \oplus \mathrm{Diag}\,\{\lambda^2 - 2\alpha_i\lambda + \alpha_i^2 + \beta_i^2\}$.
- $n(\alpha) \leq p(\alpha) \Rightarrow Q(\lambda)$ diagonalizable.

Positive Mass Matrix: Example I

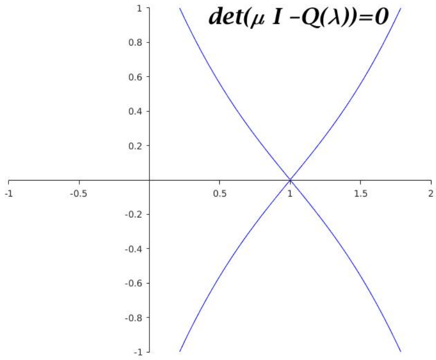
$$J = \text{Diag}\left(1, 1, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right)$$

For any choice of sign characteristic $p(1) < n(1)$.

No $Q(\lambda)$ with $M > 0$ and Jordan matrix J

Symmetric $Q(\lambda)$:

$$Q(\lambda) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \lambda^2 + \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \lambda + \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$



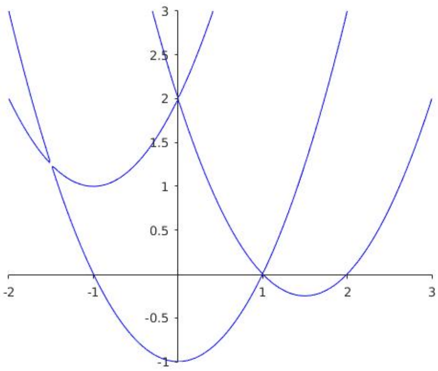
Positive Mass Matrix: Example II

$$J = \text{Diag}\left(-1, 1, 1, 2, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right)$$

Eigenvalues	-1	1	1	2	$1+i$	$1-i$
Sign Characteristic	-1	-1	$+1$	$+1$		

$p(\alpha) \geq n(\alpha)$ for all $\alpha \in [-1, 2]$

$$\begin{aligned} Q(\lambda) &= \text{Diag}((\lambda+1)(\lambda-1), (\lambda-1)(\lambda-2), \lambda^2 - 2\lambda + 2) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda^2 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \lambda + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \end{aligned}$$



Behaviour of the Eigenvalue Functions Near Zero

$L(\lambda) = L_\ell \lambda^\ell + L_{\ell-1}\lambda^{\ell-1} + \cdots + L_0$ symmetric with $\det L_\ell \neq 0$. $\mu_1(\lambda), \ldots, \mu_n(\lambda)$ its eigenvalue functions and λ_z^+, λ_z^- the positive and negative real eigenvalues of $L(\lambda)$ closest to zero. If (π, ν, δ) is the inertia of L_0 then for $\lambda \in (\lambda_z^-, \lambda_z^+)$, π eigenvalue functions are positive ($\mu_j(\lambda) > 0$) and ν are negative ($\mu_j(\lambda) < 0$).

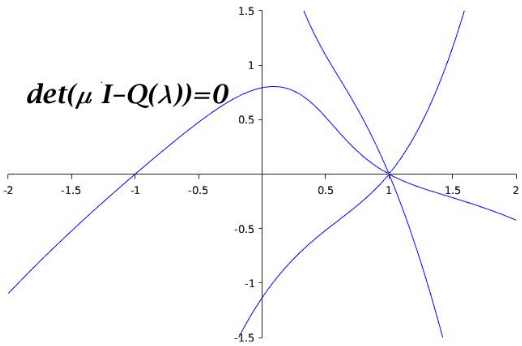
If $L_0 > 0$ then for $\lambda_z^- < \lambda < \lambda_z^+$, $\mu_j(\lambda) > 0$, $j = 1, \ldots, n$

If $L_0 \geq 0$ then $\begin{cases} \mu_j(\lambda) > 0 & \text{for } \lambda_z^- < \lambda < \lambda_z^+, \quad j = 1, \ldots, \pi \\ \mu_j(0) = 0 & \quad j = \pi + 1, \ldots, n \end{cases}$

Example II

$$Q(\lambda) = \begin{bmatrix} -0.5 & 2 & 0.5 \\ 2 & 1.4 & -0.5 \\ 0.5 & -0.5 & -0.5 \end{bmatrix} \lambda^2 + \begin{bmatrix} -1.0 & -3.7 & 0 \\ -3.7 & -1.8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \lambda + \begin{bmatrix} 1.5 & 1.7 & -0.5 \\ 1.7 & 0.4 & 0.5 \\ -0.5 & 0.5 & 0.5 \end{bmatrix}$$

$$J = \mathrm{Diag}\left(-1, 1, 1, 1, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right)$$



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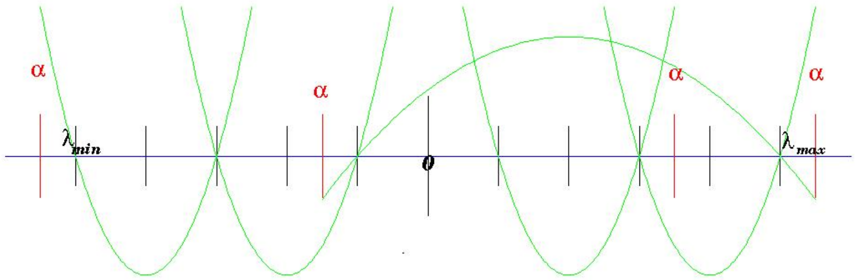
The Sign Characteristic when $L_0 > 0$

$L(\lambda) = L_\ell \lambda^\ell + L_{\ell-1}\lambda^{\ell-1} + \cdots + L_0$ symmetric and **semisimple** with $\det L_\ell \neq 0$ and $L_0 > 0$.

- For $\alpha < 0$,
    - $p_-(\alpha)$ = number of real e.v. of $L(\lambda)$ of positive type in $(\alpha, 0]$
    - $n_-(\alpha)$ = number of real e.v. of $L(\lambda)$ of negative type in $[\alpha, 0)$
- For $\alpha > 0$
    - $p_+(\alpha)$ = number of real e.v. of $L(\lambda)$ of positive type in $(0, \alpha]$
    - $n_+(\alpha)$ = number of real e.v. of $L(\lambda)$of negative type in $[0, \alpha)$.

Then $n_-(\alpha) \leq p_-(\alpha)$ for $\alpha < 0$ and $n_+(\alpha) \geq p_+(\alpha)$ for $\alpha > 0$.



The IRSQEP with Positive Definite Stiffness Matrix

There exists a semisimple real symmetric $Q(\lambda) = M\lambda^2 + D\lambda + K$ with $K > 0$ and Jordan matrix J if and only if the sign characteristic can be assigned such that

$$n_-(\alpha) \leq p_-(\alpha), \quad \text{for } \alpha < 0$$

and

$$n_+(\alpha) \geq p_+(\alpha) \quad \text{for } \alpha > 0$$

Moreover, under this conditions $Q(\lambda)$ can be constructed diagonal .

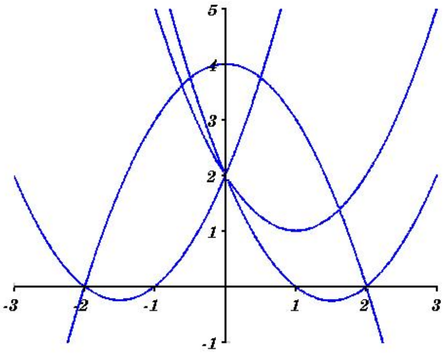
Positive Stiffness Matrix: Example

$$J = \mathrm{Diag}\left(-2, -2, -1, 1, 2, 2, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right)$$

Eigenvalues	-2	-2	-1	1	2	2	$1+i$	$1-i$
Sign Characteristic	$+1$	-1	$+1$	-1	$+1$	-1		

$$n_-(\alpha) \le p_-(\alpha),\ \alpha < 0,\ \ n_+(\alpha) \ge p_+(\alpha),\ \alpha > 0$$

$$\begin{aligned} Q(\lambda) &= \mathrm{Diag}\left((\lambda+2)(\lambda+1), -(\lambda-2)(\lambda+2), (\lambda-1)(\lambda-2), \lambda^2 - 2\lambda + 2\right) \\ &= \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\lambda^2 + \begin{bmatrix} 3 & & & \\ & 0 & & \\ & & -3 & \\ & & & -2 \end{bmatrix}\lambda + \begin{bmatrix} 2 & & & \\ & 4 & & \\ & & 2 & \\ & & & 2 \end{bmatrix} \end{aligned}$$



The IRSQEP with Positive Semidefinite Coefficients

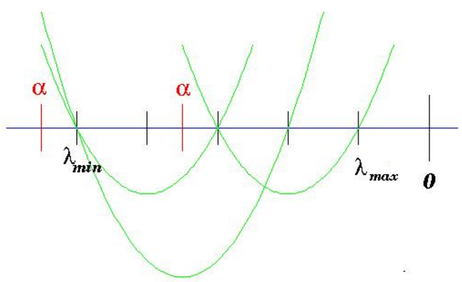
Assume

- $Q(\lambda) = M\lambda^2 + D\lambda + K$ real and symmetric,
- $M > 0$ and $K \ge 0$, and
- $\lambda_0 \in \Lambda(Q)$

$$Q(\lambda_0)x_0 = 0 \Rightarrow x_0^T Q(\lambda_0)x_0 = 0 \Rightarrow x_0^T M x_0 \lambda_0^2 + x_0^T D x_0 \lambda_0 + x_0^T K x_0 = 0$$

If $D \ge 0$ then $\mathrm{Re}\,\lambda_0 \le 0$.

There exists a semisimple real symmetric $Q(\lambda) = M\lambda^2 + D\lambda + K$ with $M > 0$, $D \ge 0$, $K > 0$ and Jordan matrix J if and only if $\mathrm{Re}\,J < 0$ and the sign characteristic can be assigned such that $n_-(\alpha) \le p_-(\alpha)$ for $\alpha < 0$ and $n_-(\lambda_{\min}) = p_-(\lambda_{\min})$.



$$\begin{aligned} Q(\lambda) &= \mathrm{Diag}\left\{(\lambda - \lambda_{2i-1})(\lambda - \lambda_{2i})\right\} \\ &\quad \oplus \mathrm{Diag}\left\{\lambda^2 - 2\alpha_i\lambda + \alpha_i^2 + \beta_i^2\right\} \end{aligned}$$

Summarizing: Admissible spectral data

From now on:

$$J = \text{Diag}\left(R_+, R_-, \begin{bmatrix} A & -B \\ B & A \end{bmatrix}\right)$$

where

- $R_+ = \text{Diag}(r_1, \ldots, r_q)$ prescribed to be of positive type.
- $R_- = \text{Diag}(r_{q+1}, \ldots, r_{2q})$ prescribed to be of negative type.
- $R_+ - R_- > 0$ and $r_j < 0$
- $A = \text{Diag}(\alpha_1, \ldots, \alpha_s)$, $B = \text{Diag}(\beta_1, \ldots, \beta_s)$, $\alpha_j < 0$, $\beta_j > 0$.
- $\alpha_j - i\beta_j, \alpha_j + i\beta_j$= prescribed non-real eigenvalues.
- $n = q + s$

Then

$$\begin{aligned} Q(\lambda) &= \text{Diag}((\lambda - r_1)(\lambda - r_{q+1}), \ldots, (\lambda - r_q)(\lambda - r_{2q})) \\ &\quad \oplus \text{Diag}(\lambda^2 - 2\alpha_1\lambda + \alpha_1^2 + \beta_1^2, \ldots \lambda^2 - 2\alpha_s\lambda + \alpha_s^2 + \beta_s^2) \end{aligned}$$

real, quadratic, symmetric with positive definite coefficients.

> Can we construct non-diagonal real symmetric quadratic matrix polynomial with positive definite coefficients and Jordan matrix J?

Selfadjoint Jordan triples

Given J, (X, J, Y) with $X \in \mathbb{R}^{n\times 2n}$ and $Y \in \mathbb{R}^{2n\times n}$ is **selfadjoint Jordan triple** if

- $\begin{bmatrix} X \\ XJ \end{bmatrix}$ is invertible
- $XY = 0$ and XJY is invertibe.
- There is a real nonsingular *symmetric* matrix H such that

$$Y = H^{-1}X^T \text{ and } J^T H = HJ$$

Why are they important?

If

$$P_k := XJ^kY = XJ^kH^{-1}X^T \quad (\textit{moments})$$

$$M = P_1^{-1}, \quad D = -P_1^{-1}P_2P_1^{-1}, \quad K = -P_1^{-1}(P_3 + P_2P_1^{-1}P_2)P_1^{-1}$$

then $Q(\lambda) = M\lambda^2 + D\lambda + K$ is real and symmetric and has J as Jordan matrix.

Alternatively $K = -P_{-1}^{-1}$ if $0 \notin \Lambda(J)$

A Convenient Selfadjoint Jordan triple

$Q(\lambda) = M\lambda^2 + D\lambda + K$ real and symmetric.

Positive type — Negative type — $\beta_i > 0$

$$\Lambda(Q) = (r_1, \dots, r_q, \; r_{q+1}, \dots, r_{2q}, \; \alpha_1 \pm i\beta_1, \dots, \alpha_s \pm i\beta_s)$$

$\mathrm{Diag}(\alpha_1, \dots, \alpha_s)$ $\quad$ $\mathrm{Diag}(\beta_1, \dots, \beta_s)$

$$J = \mathrm{Diag}\left(R_+, R_-, \begin{bmatrix} A & -B \\ B & A \end{bmatrix}\right)$$

$$P = \mathrm{Diag}(I_q, -I_q, -I_s, I_s), \qquad X = \begin{bmatrix} X_+ & X_- & V & U \end{bmatrix}$$

(X, J, PX^T) is a **real** seladjoint Jordan triple of $Q(\lambda)$

And conversely: If (X, J, PX^T) is a selfadjoint triple then it defines a **unique** real and symmetric $Q(\lambda)$.

- X_+, X_-= eigenvectors of R_+ and R_-
- U, V= real and imaginary parts of eigenvectors of $\alpha_k \pm i\beta_k$.

Recall: $XPX^T = 0$ and $XJPX^T = M^{-1}$

The Orthogonality Property

$X = \begin{bmatrix} X_+ & X_- & V & U \end{bmatrix}$, $P = \mathrm{Diag}(I_q, -I_q, -I_s, I_s)$

$$XPX^T = 0 \quad \Leftrightarrow \quad \begin{bmatrix} X_+ & U \end{bmatrix} \begin{bmatrix} X_+^T \\ U^T \end{bmatrix} = \begin{bmatrix} X_- & V \end{bmatrix} \begin{bmatrix} X_-^T \\ V^T \end{bmatrix}$$

$$\overset{(AA^T = BB^T \Leftrightarrow B = A\Theta)}{\Longleftrightarrow} \quad \boxed{\begin{bmatrix} X_- & V \end{bmatrix} = \begin{bmatrix} X_+ & U \end{bmatrix} \Theta}$$

$\Theta \in \mathbb{R}^{n \times n}$ **orthogonal** and

$\mathrm{rank}\, X = n \Leftrightarrow \det \begin{bmatrix} X_+ & U \end{bmatrix} \neq 0, \det \begin{bmatrix} X_- & V \end{bmatrix} \neq 0$

Given (J, P), (X, J, PX^T) real selfadjoint Jordan triple (of some symmetric quadratic matrix polynomials) if and only if $\begin{bmatrix} X_+ & U \end{bmatrix}$ non-singular and there is orthogonal Θ such that $\begin{bmatrix} X_- & V \end{bmatrix} = \begin{bmatrix} X_+ & U \end{bmatrix} \Theta$.

In that case (recall $P_k = XJ^kPX^T$ and assume $0 \notin \Lambda(J)$)

$$M = P_1^{-1} \quad D = -P_1^{-1}P_2P_1^{-1} \quad K = -P_{-1}^{-1}$$

The moments. Part I

$X = \begin{bmatrix} X_+ & X_- & V & U \end{bmatrix}$ and $\begin{bmatrix} X_- & V \end{bmatrix} = \begin{bmatrix} X_+ & U \end{bmatrix} \Theta$, Θ orthogonal

$$\begin{array}{c} P_1 \\ (XJPX^T) \end{array} = \begin{bmatrix} X_+ & U \end{bmatrix} \underbrace{\begin{bmatrix} I_n & \Theta \end{bmatrix} \begin{bmatrix} R_+ & & & \\ & A_1 & & -B_1 \\ & & -R_- & \\ & -B_1 & & -A_1 \end{bmatrix} \begin{bmatrix} I_n \\ \Theta^T \end{bmatrix}}_{H_1(\Theta)} \begin{bmatrix} X_+^T \\ U^T \end{bmatrix}$$

$A_1 = A, \quad B_1 = B$

- $M = P_1^{-1} > 0 \Leftrightarrow H_1(\Theta) > 0$

$$\begin{array}{c} P_2 \\ (XJ^2PX^T) \end{array} = \begin{bmatrix} X_+ & U \end{bmatrix} \underbrace{\begin{bmatrix} I_n & \Theta \end{bmatrix} \begin{bmatrix} R_+^2 & & & \\ & A_2 & & -B_2 \\ & & -R_-^2 & \\ & -B_2 & & -A_2 \end{bmatrix} \begin{bmatrix} I_n \\ \Theta^T \end{bmatrix}}_{H_2(\Theta)} \begin{bmatrix} X_+^T \\ U^T \end{bmatrix}$$

$A_2 = A^2 - B^2, \quad B_2 = 2AB$

- $D = -P_1^{-1} P_2 P_1^{-1} \geq 0 \Leftrightarrow H_2(\Theta) \leq 0$

The moments. Part II

$$P_{-1} = XJ^{-1}PX^T =$$

$$\begin{bmatrix} X_+ & U \end{bmatrix} \underbrace{\begin{bmatrix} I_n & \Theta \end{bmatrix} \begin{bmatrix} R_+^{-1} & & & \\ & A_{-1} & & B_{-1} \\ & & -R_-^{-1} & \\ & B_{-1} & & -A_{-1} \end{bmatrix} \begin{bmatrix} I_n \\ \Theta^T \end{bmatrix}}_{H_{-1}(\Theta)} \begin{bmatrix} X_+^T \\ U^T \end{bmatrix}$$

$$A_{-1} = \mathrm{Diag}\left\{ \frac{\alpha_k}{|\alpha_k \pm i\beta_k|^2} \right\}, \quad B_{-1} = \mathrm{Diag}\left\{ \frac{\beta_k}{|\alpha_k \pm i\beta_k|^2} \right\}$$

- $K = -P_{-1}^{-1} > 0 \Leftrightarrow H_{-1}(\Theta) < 0$

Characterize the orthogonal matrices Θ such that $H_1(\Theta) > 0$, $H_2(\Theta) < 0$ and $H_{-1}(\Theta) < 0$

Back to Diagonal Systems

Recall the conditions of admissible spectral data:

- $R_+ - R_- > 0$,
- $r_j < 0$, $\alpha_j < 0$, $\beta_j > 0$.

If

$$\Theta_0 = \begin{bmatrix} I_q & 0 \\ 0 & -I_s \end{bmatrix}, [X_+ \quad U] = \begin{bmatrix} (R_+ - R_-)^{-1/2} & 0 \\ 0 & (2B)^{-1/2} \end{bmatrix}$$

then

$$H_1(\Theta_0) = \begin{bmatrix} R_+ - R_- & 0 \\ 0 & 2B \end{bmatrix} > 0, \qquad H_2(\Theta_0 = \begin{bmatrix} R_+^2 - R_-^2 & 0 \\ 0 & 4AB \end{bmatrix} < 0$$

$$H_{-1}(\Theta_0) = \begin{bmatrix} R_+^{-1} - R_-^{-1} & 0 \\ 0 & -B_{-1} \end{bmatrix} < 0$$

and

$$M = P_1^{-1} = I_n > 0, \quad D = -P_1^{-1} P_2 P_1^{-1} = \begin{bmatrix} -(R_+ + R_-), & 0 \\ 0 & -2M \end{bmatrix} > 0,$$

$$K = -P_1^{-1} = \begin{bmatrix} (R_+ R_-) & 0 \\ 0 & \text{Diag}\left\{|\alpha_k \pm i\beta_k|^2\right\} \end{bmatrix} > 0.$$

Open sets

$$H_i(\Theta) = \begin{bmatrix} I_n & \Theta \end{bmatrix} \begin{bmatrix} R_+^i & & & \\ & A_i & & B_i \\ & & -R_-^i & \\ & B_i & & -A_i \end{bmatrix} \begin{bmatrix} I_n \\ \Theta^T \end{bmatrix}, \quad i = -1, 1, 2.$$

$$\mathcal{P}_1 = \{\Theta : H_1(\Theta) > 0\}, \qquad \mathcal{P}_2 = \{\Theta : H_2(\Theta) < 0\},$$
$$\mathcal{P}_{-1} = \{\Theta : H_{-1}(\Theta) < 0\}$$

are open sets in the orthogonal group and

$$\Theta_0 = \begin{bmatrix} I_q & 0 \\ 0 & -I_s \end{bmatrix} \in \mathcal{P}_1 \cap \mathcal{P}_2 \cap \mathcal{P}_{-1}$$

Any orthogonal Θ close enough to Θ_0 will be in $\mathcal{P}_1 \cap \mathcal{P}_2 \cap \mathcal{P}_{-1}$

A Procedure for Constructing Non-diagonal Matrices

A procedure proposed in Lancaster,Z. (2014):

For (J, P) admissible with $0 \notin \Lambda(J)$

1. Take $\Theta_0 = \mathrm{Diag}(I_q, -I_s)$.
2. Take an orthogonal Θ "close enough" to $\widetilde{\Theta}$ and/or any nonsingular $X_1 = \begin{bmatrix} X_+ & U \end{bmatrix}$.
3. Compute $H_i(\Theta)$ for $i = -1, 1, 2$.
4. Compute the moments $P_i = X_1 H_i(\Theta) X_1^T$ for $i = -1, 1, 2$.
5. Compute $M = P_1^{-1}$, $D = -P_1^{-1} P_2 P_1^{-1}$, $K = -P_{-1}^{-1}$

- $Q(\lambda) = M\lambda^2 + D\lambda + K$ is the quadratic matrix polynomial of a semisimple vibrating system with J as Jordan matrix and P as sign characteristic.
- If $\begin{bmatrix} X_- & V_- \end{bmatrix} = \begin{bmatrix} X_+ & U \end{bmatrix} \Theta$ then:
  - the columns of $\begin{bmatrix} X_+ & X_- \end{bmatrix}$ are eigenvectors of $Q(\lambda)$ associated to the real eigenvalues and
  - the columns of $\begin{bmatrix} U & V \end{bmatrix}$ are the real and imaginary parts of eigenvectors associated to the non-real eigenvalues.

MATLAB Code

Eigenvalues	-3	-2	-2	$-3/2$	-1	-1	$-2 \pm i$	$-1 \pm 2i$
Sign Characteristic	-1	-1	$+1$	-1	$+1$	$+1$		

MATLAB

```
>> Rp=diag([ -2 -1 -1]); Rm=diag([-3 -2 -3/2]); A=diag([-2 -1]); B=diag([1 2]);
>> Z32=zeros(3,2); Z23=zeros(2,3); Z33=zeros(3); Z22=zeros(2);
>> K1=[Rp Z32 Z33 Z32; Z23 A Z23 -B; Z33 Z32 -Rm Z32; Z23 -B Z23 -A];
>> A2=A^2-B^2; B2=2*A*B;
>> K2=[Rp^2 Z32 Z33 Z32; Z23 A2 Z23 -B2; Z33 Z32 -Rm^2 Z32; Z23 -B2 Z23 -A2];
>> Am1=diag([-2/5 -1/5]); Bm1=diag([1/5 2/5]);
>> Km1=[Rp^(-1) Z32 Z33 Z32; Z23 Am1 Z23 Bm1; Z33 Z32 -Rm^(-1) Z32; Z23 Bm1 Z23 -Am1];
>> Th0=blkdiag(eye(3),-eye(2));
>> T0=[eye(5) Th0];
>> H10=T0*K1*T0', H20=T0*K2*T0',Hm10=T0*Km1*T0';
>> X0=blkdiag(sqrt((Rp-Rm)^(-1)), sqrt((2*N)^(-1)));
>> P1=X0*H10*X0',P2=X0*H20*X0', Pm1=X0*Hm10*X0'
>> M=P1^(-1), D=-P1^(-1)*P2*P1^(-1), K=-Pm1^(-1)
M =
    1.0000         0         0         0         0
         0    1.0000         0         0         0
         0         0    1.0000         0         0
         0         0         0    1.0000         0
         0         0         0         0    1.0000
D =
    5.0000         0         0         0         0
         0    3.0000         0         0         0
         0         0    2.5000         0         0
         0         0         0    4.0000         0
         0         0         0         0    2.0000
K =
    6.0000         0         0         0         0
         0    2.0000         0         0         0
         0         0    1.5000         0         0
         0         0         0    5.0000         0
         0         0         0         0    5.0000
```

MATLAB Code

>>Tht=Th0+randn(5)*10^(-6); [Tht]=qr(Tht);

```
>> [I]=find(S);
>> R=1:5; R(I)=[];
>> S(I)=-ones(length(I),1); S(R)=ones(length(R),1); S=diag(S);
>> Tht=S*Tht;
>> norm(Th0-Tht)
ans =
   1.9421e-06
>> T=[eye(5) Tht];
>> H1=T*K1*T'; H2=T*K2*T';Hm1=T*Km1*T';
```

>> X1=randn(5);

```
>> P1=X1*H1*X1';P2=X1*H2*X1';Pm1=X1*Hm1*X1';
>> [eig(P1) eig(P2) eig(Pm1)]
ans =
   27.8146  -66.8082   -0.0046
   14.5485  -49.7103   -1.8946
    0.0235   -0.1001   -0.7679
    3.3399   -9.6053   -4.0538
    2.1873   -6.5207   -5.6748
```


MATLAB Code

```
>>  M=P1^(-1), D=-P1^(-1)*P2*P1^(-1), K=-Pm1^(-1)
M =
    6.3175   -9.3610 -10.3194   -3.9092   -3.2760
   -9.3610   14.9020  16.0147    5.9497    5.2358
  -10.3194   16.0147  17.4931    6.4714    5.6193
   -3.9092    5.9497   6.4714    2.8564    2.0970
   -3.2760    5.2358   5.6193    2.0970    1.8823

D =
   27.7707 -42.2651 -45.7524 -18.4179 -15.0638
  -42.2651  67.0259  71.5898  28.4607  23.9506
  -45.7524  71.5898  77.2474  30.6699  25.5605
  -18.4179  28.4607  30.6699  13.7393  10.2471
  -15.0638  23.9506  25.5605  10.2471   8.6535

K =
   31.0939 -48.0182 -51.2242 -21.5476 -17.1735
  -48.0182  75.9431  80.4497  33.7424  27.0882
  -51.2242  80.4497  85.8520  36.0527  28.7672
  -21.5476  33.7424  36.0527  16.4910  12.0532
  -17.1735  27.0882  28.7672  12.0532   9.8611
```

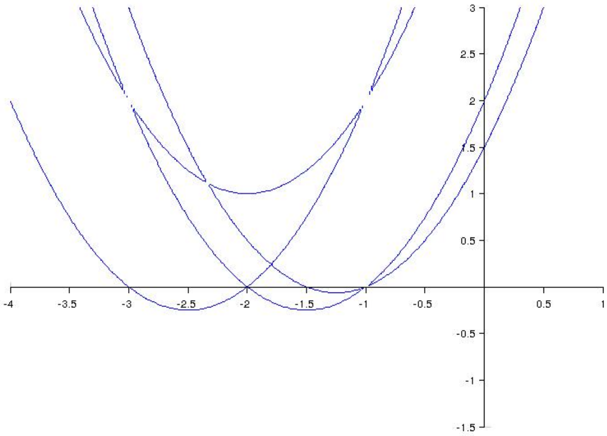
>> [eig(M) eig(D) eig(K)]

```
ans =
   42.5902  191.8295  216.9882
    0.4572    1.4796    1.3022
    0.2994    0.7994    0.5278
    0.0687    0.2528    0.2467
    0.0360    0.0753    0.1762
```

Appendix:Diagonal matrix

Eigenvalues	-3	-2	-2	$-3/2$	-1	-1	$-2 \pm i$	$-1 \pm 2i$
Sign Characteristic	-1	-1	$+1$	-1	$+1$	$+1$		

$$M = \begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
$$D = \begin{bmatrix} 5.0 & 0 & 0 & 0 & 0 \\ 0 & 3.0 & 0 & 0 & 0 \\ 0 & 0 & 2.5 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$
$$K = \begin{bmatrix} 6.0 & 0 & 0 & 0 & 0 \\ 0 & 2.0 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$



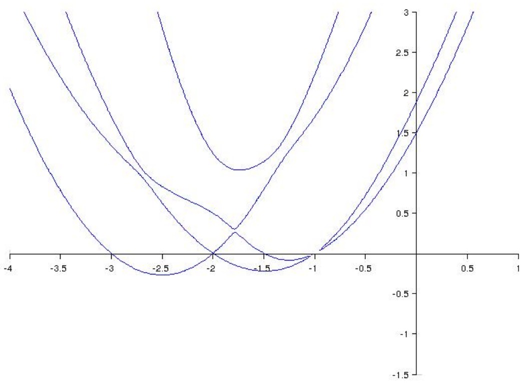
$e.v.:$

M	D	K
1	5.0	6
1	3.0	2
1	2.5	1.5
1	4	5
1	2	5

Appendix: Perturbation on X_0 and Θ_0

Eigenvalues	-3	-2	-2	$-3/2$	-1	-1	$-2 \pm i$	$-1 \pm 2i$
Sign Characteristic	-1	-1	$+1$	-1	$+1$	$+1$		

$$M = \begin{bmatrix} 1.12 & -0.16 & 0.08 & 0.13 & 0.11 \\ -0.16 & 0.88 & 0.23 & 0.17 & -0.14 \\ 0.08 & 0.23 & 1.76 & 0.83 & 0.18 \\ 0.13 & 0.17 & 0.83 & 1.34 & 0.35 \\ 0.11 & -0.14 & 0.18 & 0.35 & 1.21 \end{bmatrix}$$
$$D = \begin{bmatrix} 5.50 & -0.49 & 0.56 & 0.73 & 0.16 \\ -0.49 & 2.63 & 0.56 & 0.45 & -0.47 \\ 0.56 & 0.56 & 4.50 & 2.43 & 0.31 \\ 0.73 & 0.45 & 2.43 & 4.87 & 0.61 \\ 0.16 & -0.47 & 0.31 & 0.61 & 2.47 \end{bmatrix}$$
$$K = \begin{bmatrix} 6.57 & -0.28 & 0.84 & 1.04 & 0.43 \\ -0.28 & 1.75 & 0.37 & 0.34 & -0.14 \\ 0.84 & 0.37 & 3.09 & 2.45 & 1.13 \\ 1.04 & 0.34 & 2.45 & 6.19 & 1.87 \\ 0.43 & -0.14 & 1.13 & 1.87 & 5.95 \end{bmatrix}$$



$e.v.:$

M	D	K
2.6	7.7	9.7
1.4	5.2	6.1
1.0	3.0	4.4
0.6	2.8	1.9
0.7	2.2	1.5

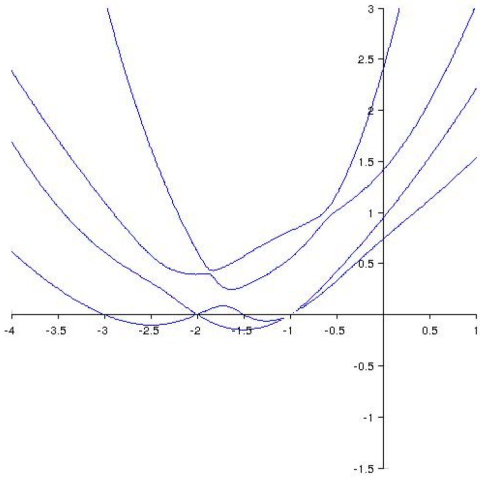
Appendix: Increasing the perturbation on X_0 and Θ_0

Eigenvalues	-3	-2	-2	$-3/2$	-1	-1	$-2 \pm i$	$-1 \pm 2i$
Sign Characteristic	-1	-1	$+1$	-1	$+1$	$+1$		

$$M = \begin{bmatrix} 0.53 & 0.27 & -0.53 & -0.31 & -0.53 \\ 0.27 & 0.90 & -0.54 & -0.71 & -0.81 \\ -0.53 & -0.54 & 1.38 & 0.35 & 1.12 \\ -0.31 & -0.71 & 0.35 & 1.45 & 0.80 \\ -0.53 & -0.81 & 1.12 & 0.80 & 1.45 \end{bmatrix}$$

$$D = \begin{bmatrix} 2.36 & 1.10 & -1.94 & -1.80 & -2.42 \\ 1.10 & 2.80 & -1.66 & -2.55 & -2.62 \\ -1.94 & -1.66 & 3.94 & 1.75 & 3.37 \\ -1.80 & -2.55 & 1.75 & 5.50 & 3.07 \\ -2.42 & -2.62 & 3.37 & 3.07 & 4.57 \end{bmatrix}$$

$$K = \begin{bmatrix} 2.76 & 1.13 & -1.80 & -2.12 & -2.37 \\ 1.13 & 2.09 & -1.45 & -2.46 & -2.38 \\ -1.80 & -1.45 & 3.17 & 2.60 & 3.25 \\ -2.12 & -2.46 & 2.60 & 6.43 & 4.09 \\ -2.37 & -2.38 & 3.25 & 4.09 & 5.53 \end{bmatrix}$$



$e.v.:$

M	D	K
3.7709	13.2424	14.4991
1.1640	3.2634	2.4009
0.3829	1.3591	1.4125
0.2419	0.4326	0.9445
0.1874	0.8992	0.7380

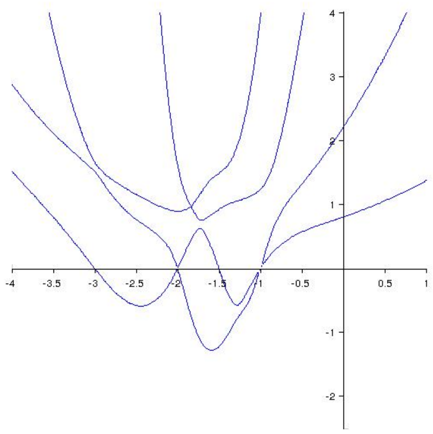
Appendix: Random X_1 with original Θ_0

Eigenvalues	-3	-2	-2	$-3/2$	-1	-1	$-2 \pm i$	$-1 \pm 2i$
Sign Characteristic	-1	-1	$+1$	-1	$+1$	$+1$		

$$M = \begin{bmatrix} 47.28 & 19.59 & -1.11 & 39.20 & -50.19 \\ 19.59 & 19.84 & 2.46 & 21.10 & -18.90 \\ -1.11 & 2.46 & 1.78 & 1.34 & 1.38 \\ 39.20 & 21.10 & 1.34 & 37.53 & -41.83 \\ -50.19 & -18.90 & 1.38 & -41.83 & 54.28 \end{bmatrix}$$

$$D = \begin{bmatrix} 134.22 & 62.45 & 3.46 & 122.50 & -144.58 \\ 62.45 & 65.30 & 12.07 & 72.67 & -61.38 \\ 3.46 & 12.07 & 7.37 & 12.36 & -3.55 \\ 122.50 & 72.67 & 12.36 & 131.17 & -133.58 \\ -144.58 & -61.38 & -3.55 & -133.58 & 158.81 \end{bmatrix}$$

$$K = \begin{bmatrix} 103.48 & 50.94 & 9.42 & 104.54 & -112.02 \\ 50.94 & 53.51 & 13.88 & 63.76 & -52.18 \\ 9.42 & 13.88 & 7.99 & 17.61 & -10.27 \\ 104.54 & 63.76 & 17.61 & 124.10 & -116.59 \\ -112.02 & -52.18 & -10.27 & -116.59 & 124.85 \end{bmatrix}$$



$e.v.:$

M	D	K
144.2587	443.8430	371.3219
13.5319	41.7984	31.4381
2.2717	8.9898	8.1904
0.1723	0.3559	0.8032
0.4926	1.9024	2.2054

Future work

- $M \geq 0$, $D \geq 0$, $K \geq 0$: admissible spectral data?
- $\det M = 0$ requires eigenvalue functions at ∞
- Arbitrary Jordan matrix: more involved analysis of the sign characteristic.

Thank you very much!