Matrix Completion Problems

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Universidad Nacional de Educación a Distancia

Madrid, 16th of June 2017

Definition of Partial Matrix

$$P = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & x_1 \\ \mathbf{1} & \mathbf{2} & x_2 & \mathbf{0} & x_3 \\ x_4 & x_5 & x_6 & \mathbf{1} & \mathbf{2} \\ \mathbf{1} & x_7 & \mathbf{1} & \mathbf{1} & \mathbf{2} \\ x_8 & \mathbf{0} & \mathbf{2} & x_9 & \mathbf{4} \end{bmatrix}$$

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Shorthand of Partial matrix

$$C_1 = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & 2 \\ \mathbf{1} & \mathbf{2} & 2 & \mathbf{0} & 0 \\ 2 & 2 & 3 & \mathbf{1} & \mathbf{2} \\ \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & \mathbf{2} \\ 2 & \mathbf{0} & \mathbf{2} & 2 & \mathbf{4} \end{bmatrix} \qquad C_2 = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{2} & 5 & \mathbf{0} & 1 \\ 1 & 0 & 0 & \mathbf{1} & \mathbf{2} \\ \mathbf{1} & 9 & \mathbf{1} & \mathbf{1} & \mathbf{2} \\ 7 & \mathbf{0} & \mathbf{2} & 4 & \mathbf{4} \end{bmatrix}$$

Completions

Types of matrix completion problems

Given a partial matrix, find a completion such that ....

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(A) With prescribed entries in general position

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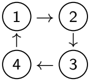
(C) the corresponding digraph is of certain type

$$\begin{bmatrix} \cdot & 5 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & 7 \\ 8 & \cdot & \cdot & \cdot \end{bmatrix} \Rightarrow$$



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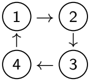
$\begin{bmatrix} \cdot & 5 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & 7 \\ 8 & \cdot & \cdot & \cdot \end{bmatrix} \Rightarrow$ 

(I) it has certain prescribed spectral property

- (II.A) invariant factors
- (II.B) eigenvalues
- (II.C) characteristic polynomial
- (II.D) rank (minimal, maximal)

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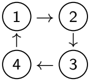

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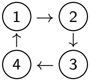

(1) with exact algorithm

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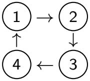

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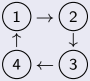

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Positive Definite matrices (Hermitian such that $\mathbf{x}^* A\mathbf{x} > 0$ for nonzero $\mathbf{x} \in \mathbb{C}^n$).

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B is a **partial positive definite** matrix if every fully prescribed principal submatrix of B is positive definite.

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A (di)graph that has a loop at every vertex has the **positive definite completion property** if and only if it is chordal[1].

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[1]chordal graph: one in which all cycles of four or more vertices have a *chord (an edge that connects two vertices of the cycle).*

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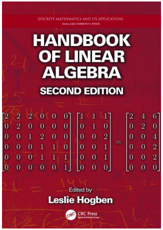
Charlie Johnson and Leslie Hogben have been champions of these type of problems.



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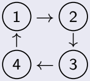

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The Problem

Let $\mathbb{F}$ be field, let $\Lambda = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{F}$, and let $P \in \overline{\mathbb{F}}^{n \times n}$ be a partial matrix.

How many entries can be prescribed so it is possible to complete P and obtain a matrix with Λ as eigenvalues?

Theorem (London-Minc)

For $n - 1$ prescribed entries or less (and any Λ) it is always possible to complete P.

Inconsistencies

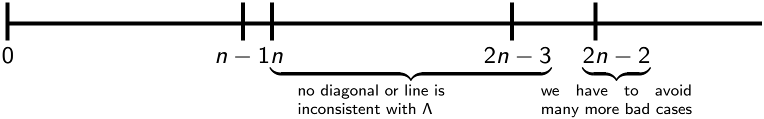
With n prescribed entries we just have to avoid two cases:

$$\begin{bmatrix} a & \cdot & \cdot & \cdot & \cdot \\ \cdot & b & \cdot & \cdot & \cdot \\ \cdot & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & d & \cdot \\ \cdot & \cdot & \cdot & \cdot & e \end{bmatrix}$$

If $a + b + c + d + e \neq \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5$, then its impossible to complete, so we say that the **diagonal is inconsistent with** Λ.

$$\begin{bmatrix} a & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

If $a \notin \Lambda$ then its impossible to complete, so we say that the corresponding **line** (row or column) **is inconsistent with** Λ.



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With *n* prescribed entries we just have to avoid two cases:

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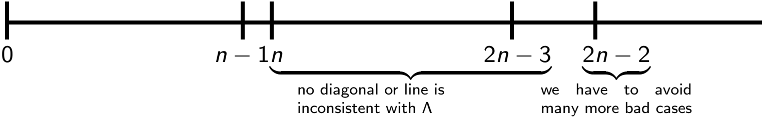

Theorem (Hershkowitz)

Let $\mathbb{F}$ be a field, and let $\Lambda = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{F}$. Let an $n \times n$ matrix P have at most $2n-3$ prescribed entries that belong to $\mathbb{F}$, and such that the diagonal and lines of P are none inconsistent with Λ.

Then P can be completed with elements of $\mathbb{F}$ to obtain a matrix with spectrum Λ.

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Matrices P with no full line	Matrices P with full line
$\downarrow$	$\downarrow$
Algorithm to complete P only using $+$ and -	Algorithm to complete P using $+$, - and $/$
$\downarrow$	$\downarrow$

Theorem (Borobia, C., Smigoc)

Hershkowitz's Thm. for partial matrices with no full line is valid for integral domains[2]

Algorithm (B, C., S.)

Easy algorithm for Hershkowitz's Thm. that completes P

2

[2] a commutative ring with identity that is a subring of a field, e.g. $\mathbb{Z}$).

Gradient Flow Method (Chu, Diele and Sgura, 2004)

We can consider the completion problem as a optimization problem. Let $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and let π be the projection onto the space of possible completions. We have to minimize:

$$g(V) = \frac{1}{2}\langle VDV^{-1} - \pi(VDV^{-1}), VDV^{-1} - \pi(VDV^{-1})\rangle$$

$$\nabla g(V) = \left[VDV^{-1} - \pi(VDV^{-1}), (VDV^{-1})^T\right] V^{-T}$$

The vector field

$$\frac{dV}{dt} = \left[(VDV^{-1})^T, VDV^{-1} - \pi(VDV^{-1})\right] V^{-T}$$

defines a flow in the open ser $GL(n)$ and moves in the steepest descent direction to reduce the value of $g(V)$.

	invariant factors	characteristic polynomial	eigenvalues	rank
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prescribed entries forming blocks				

The Problem

Let $\mathbb{F}$ be field, let $f(\lambda)$ be a monic polynomial, and let and let P be a $n \times n$ partial matrix.

How many entries can be prescribed so it is possible to complete P with characteristic polynomial $f(\lambda)$?

Theorem (Dias da Silva)

For $n-1$ prescribed entries it is always possible to complete P, **except** when

P is "**reducible**" $\begin{bmatrix} \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$ and $f(\lambda)$ has no roots in $\mathbb{F}$.

Inconsistencies

With n prescribed entries we have to avoid several more cases:

$$\begin{bmatrix} a & \cdot & \cdot & \cdot & \cdot \\ \cdot & b & \cdot & \cdot & \cdot \\ \cdot & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & d & \cdot \\ \cdot & \cdot & \cdot & \cdot & e \end{bmatrix}$$

If $-(a+b+c+d+e)$ is not the coefficient of λ^{n-1} in $f(\lambda)$, then its impossible to complete, so we say that the **diagonal is inconsistent with** $f(\lambda)$.

$$\begin{bmatrix} a & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

If $f(a) \neq 0$ then its impossible to complete, so we say that the corresponding **line** (row or column) **is inconsistent with** $f(\lambda)$.

Five small sporadic cases.

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Theorem (Zaballa)

Let $\mathbb{F}$ be a field, and let $f(\lambda) \in \mathbb{F}[\lambda]$. Let P be a $n \times n$ partial matrix with n prescribed entries, then it can always be completed to obtain a matrix with characteristic polynomial $f(\lambda)$, **except** when:

1. The diagonal or a line of P is inconsistent with $f(\lambda)$.
2. P is reducible and $f(\lambda)$ has no roots in $\mathbb{F}$.
3. Five small sporadic cases.

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<table>
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</table>

Partial matrices

Example (Completions of a Partial matrix)

$$P = \begin{bmatrix} x & z & \mathbf{1} \\ y & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & t \end{bmatrix} \quad \xrightarrow{x=y=z=t=1} \quad C_1 = \begin{bmatrix} 1 & 1 & \mathbf{1} \\ 1 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 1 \end{bmatrix} \longrightarrow \text{rank}\,1$$

$$\xrightarrow{\substack{x=y=0 \\ z=t=1}} \quad C_2 = \begin{bmatrix} 0 & 1 & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 1 \end{bmatrix} \longrightarrow \text{rank}\,2$$

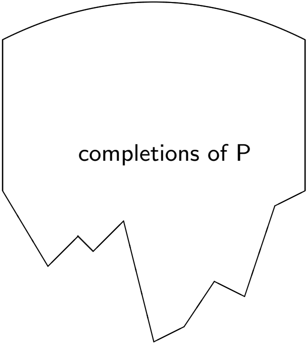
$$\xrightarrow{x=y=z=t=0} \quad C_3 = \begin{bmatrix} 0 & 0 & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 0 \end{bmatrix} \longrightarrow \text{rank}\,3$$

$\text{rank}\,P = \{1, 2, 3\}$

$\text{minRank}\,P = 1$

$\text{maxRank}\,P = 3$

$$P = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot \\ \cdot & 3 & 0 & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot & 2 \\ \cdot & \cdot & \cdot & 4 & 0 \\ \cdot & \cdot & \cdot & 5 & \cdot \end{bmatrix}$$



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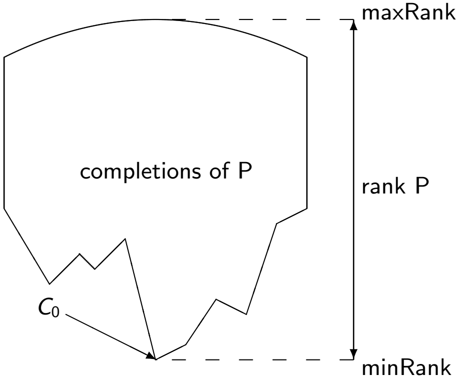

$$P = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot \\ \cdot & 3 & 0 & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot & 2 \\ \cdot & \cdot & \cdot & 4 & 0 \\ \cdot & \cdot & \cdot & 5 & \cdot \end{bmatrix}$$

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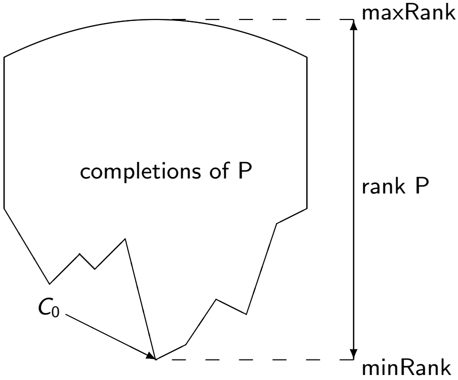
NP-hard



$$P = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot \\ \cdot & 3 & 0 & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot & 2 \\ \cdot & \cdot & \cdot & 4 & 0 \\ \cdot & \cdot & \cdot & 5 & \cdot \end{bmatrix}$$

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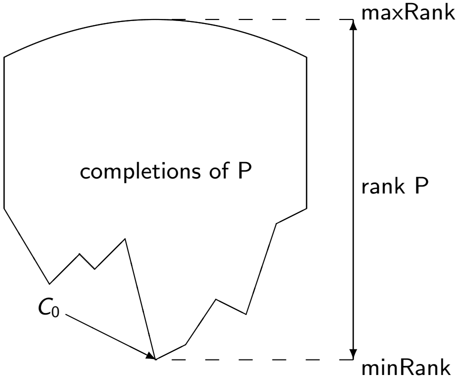
NP-hard

rank $C = \#$ nonzero singular vals of C



$$P = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot \\ \cdot & 3 & 0 & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot & 2 \\ \cdot & \cdot & \cdot & 4 & 0 \\ \cdot & \cdot & \cdot & 5 & \cdot \end{bmatrix}$$

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⇕

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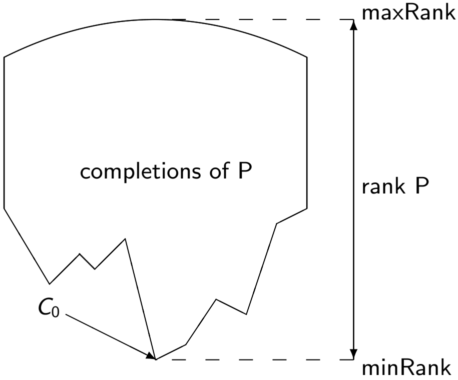

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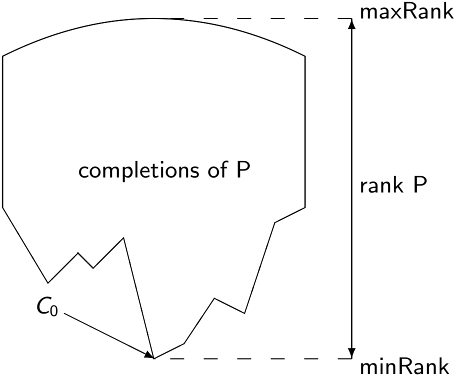
Convex relaxation method

$$\begin{cases} \text{minimize:} & ||C||_* \\ \text{subject to:} & C \text{ is completion of } P \end{cases}$$



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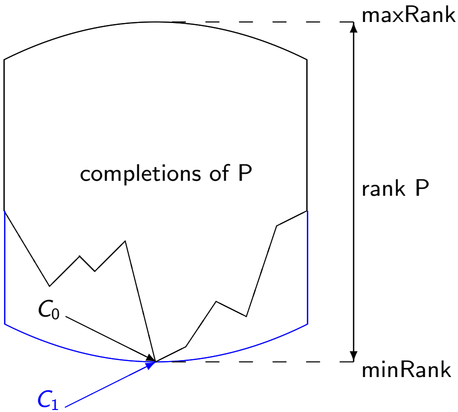

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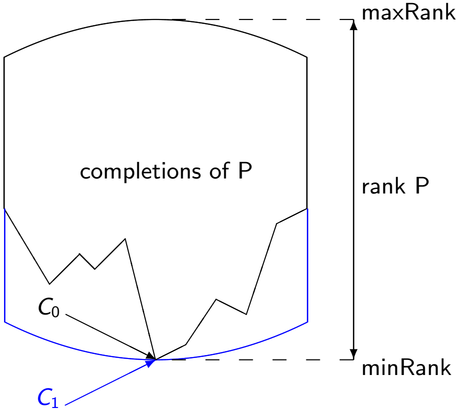
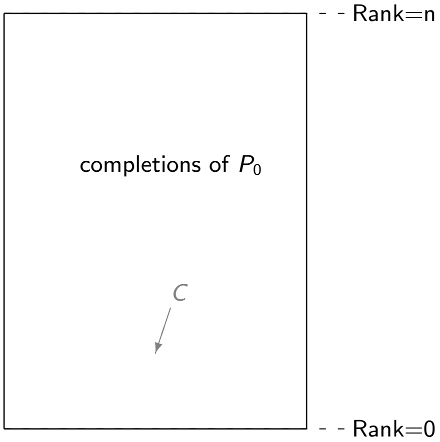
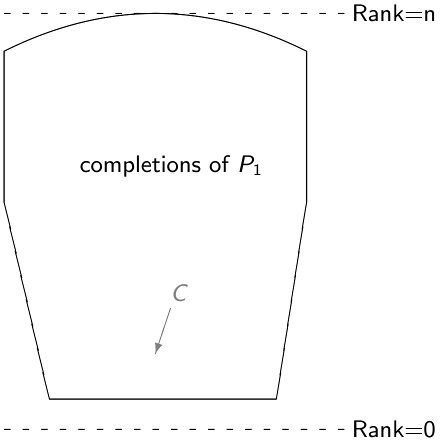
Polynomial time



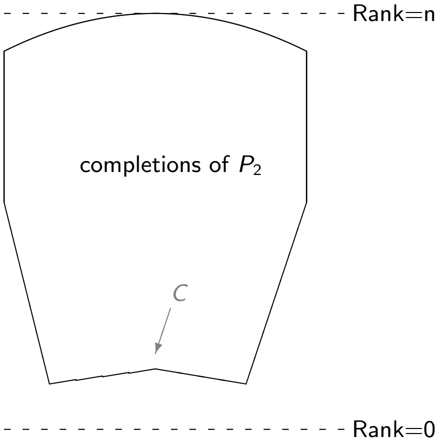
$$P_0 = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 1 & 2 & 2 & 0 & 0 \\ 2 & 2 & 3 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 \\ 2 & 0 & 2 & 2 & 4 \end{bmatrix}$$



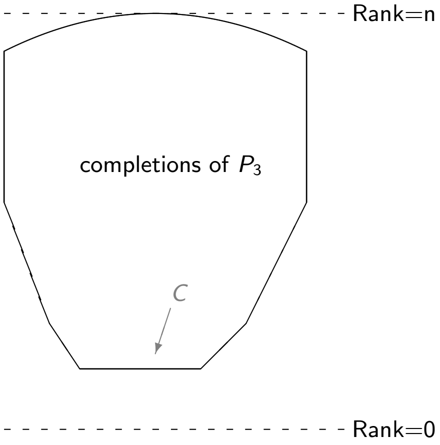
$$P_1 = \begin{bmatrix} \mathbf{1} & 0 & 1 & 1 & 2 \\ 1 & 2 & 2 & 0 & 0 \\ 2 & 2 & 3 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 \\ 2 & 0 & 2 & 2 & 4 \end{bmatrix}$$



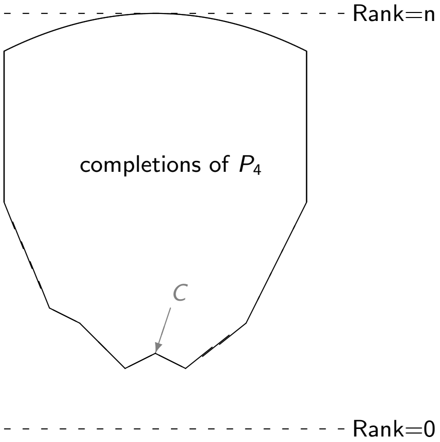
$$P_2 = \begin{bmatrix} \mathbf{1} & 0 & 1 & 1 & 2 \\ 1 & 2 & 2 & \mathbf{0} & 0 \\ 2 & 2 & 3 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 \\ 2 & 0 & 2 & 2 & 4 \end{bmatrix}$$



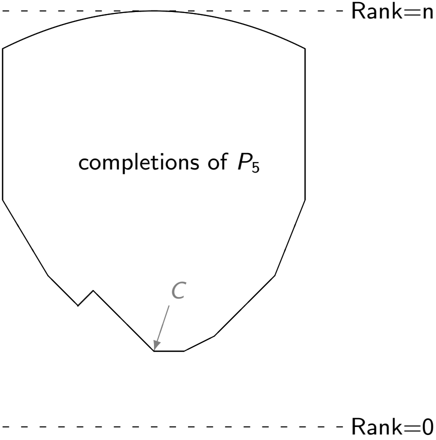
$$P_3 = \begin{bmatrix} \mathbf{1} & 0 & 1 & 1 & 2 \\ 1 & \mathbf{2} & 2 & \mathbf{0} & 0 \\ 2 & 2 & 3 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 \\ 2 & 0 & 2 & 2 & 4 \end{bmatrix}$$



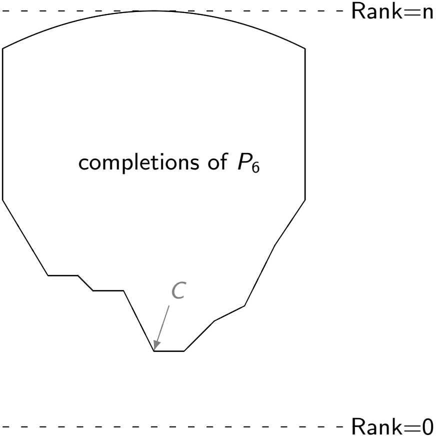
$$P_4 = \begin{bmatrix} \mathbf{1} & 0 & 1 & 1 & 2 \\ 1 & \mathbf{2} & 2 & \mathbf{0} & 0 \\ 2 & 2 & 3 & 1 & 2 \\ 1 & 0 & 1 & 1 & \mathbf{2} \\ 2 & 0 & 2 & 2 & 4 \end{bmatrix}$$



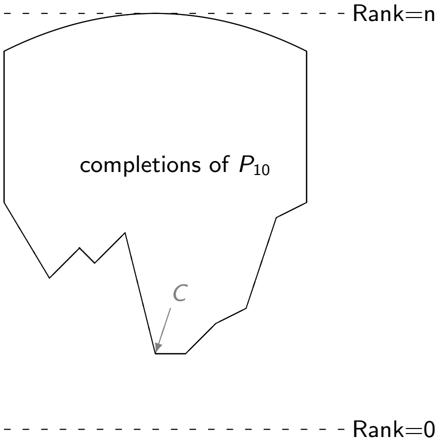
$$P_5 = \begin{bmatrix} \mathbf{1} & 0 & 1 & 1 & 2 \\ 1 & \mathbf{2} & 2 & \mathbf{0} & 0 \\ 2 & 2 & 3 & 1 & 2 \\ 1 & 0 & 1 & 1 & \mathbf{2} \\ 2 & 0 & \mathbf{2} & 2 & 4 \end{bmatrix}$$



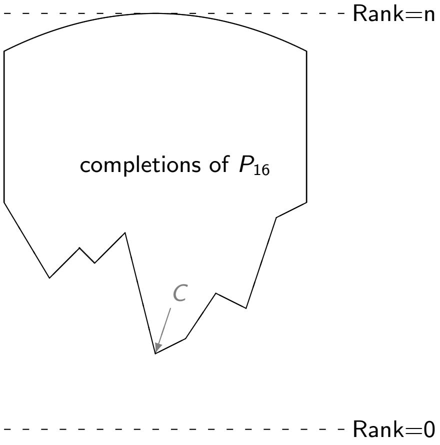
$$P_6 = \begin{bmatrix} \mathbf{1} & 0 & 1 & 1 & 2 \\ 1 & \mathbf{2} & 2 & \mathbf{0} & 0 \\ 2 & 2 & 3 & 1 & 2 \\ 1 & 0 & 1 & 1 & \mathbf{2} \\ 2 & \mathbf{0} & \mathbf{2} & 2 & 4 \end{bmatrix}$$



$$P_{10} = \begin{bmatrix} \mathbf{1} & 0 & \mathbf{1} & 1 & 2 \\ \mathbf{1} & \mathbf{2} & 2 & \mathbf{0} & 0 \\ 2 & 2 & 3 & \mathbf{1} & 2 \\ 1 & 0 & \mathbf{1} & 1 & \mathbf{2} \\ 2 & \mathbf{0} & \mathbf{2} & 2 & 4 \end{bmatrix}$$



$$P_{16} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & 2 \\ \mathbf{1} & \mathbf{2} & 2 & \mathbf{0} & 0 \\ 2 & 2 & 3 & \mathbf{1} & \mathbf{2} \\ \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & \mathbf{2} \\ 2 & \mathbf{0} & \mathbf{2} & 2 & \mathbf{4} \end{bmatrix}$$



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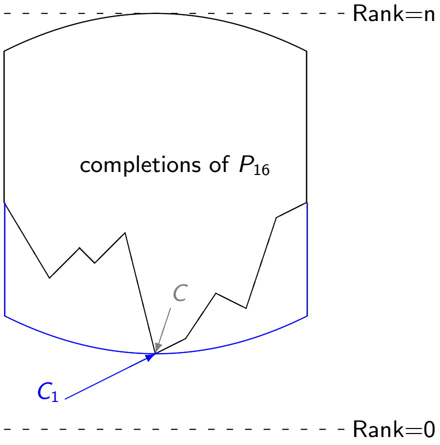

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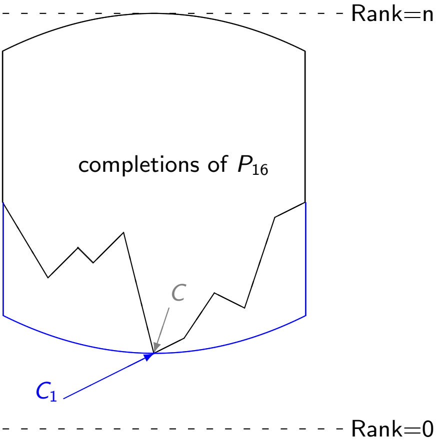

For unique recovery of a matrix of rank r we need $\geq 2nr - r^2$ observations:

rank 2 and 2×2 nonsingular: $\begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 1 & 2 & 2 & 0 & 0 \\ 2 & 2 & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & \cdot & \cdot \\ 2 & 0 & \cdot & \cdot & \cdot \end{bmatrix}$.

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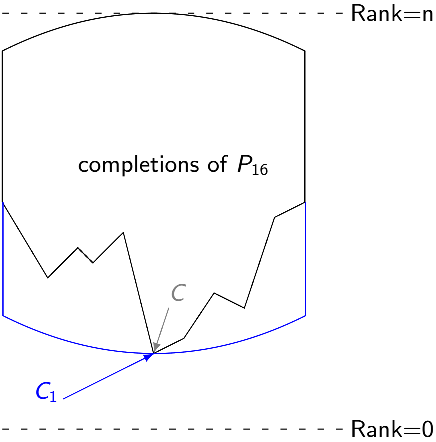

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When randomly sampled, the min number of observations is better thought of as being $nr\log(n)$ (coupon collectors effect).

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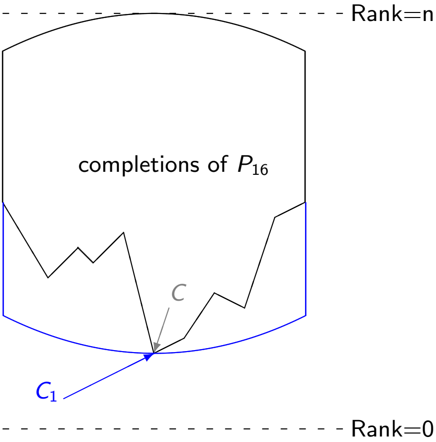

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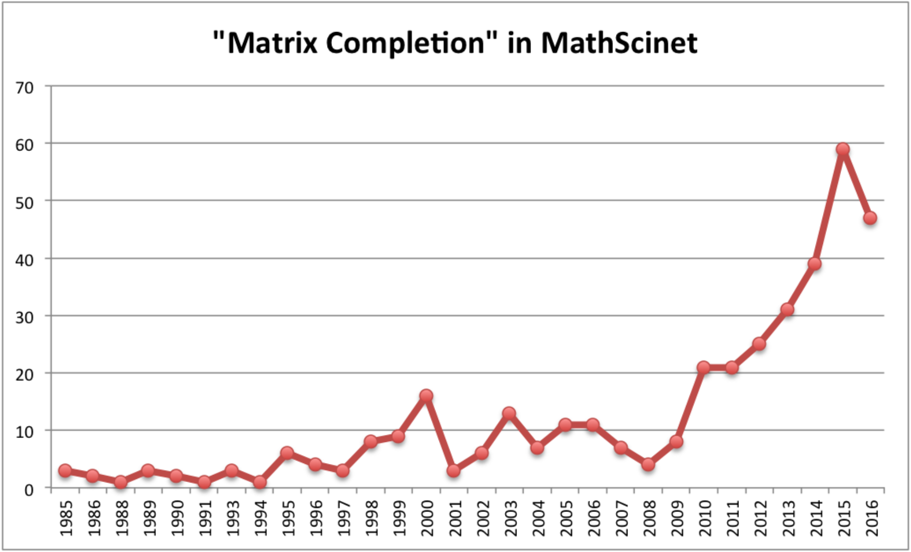
Compressed sensing (M. Fazel, B. Recht, E. Candès, T. Tao [2003, 2009, 2010])

If there are $\alpha nr\log^2(n)$ observed entries selected uniformly at random, and the singular vectors of C are *incoherent, then with very high probability the convex relaxation method yields* C*.*

Number of papers with term "Matrix Completion"



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Can linear algebra techniques say something on this or related problems?

The new frontier in minRank matrix completion is possibly Finite Fields (FF)

1. The optimization techniques (convex relaxation) only works for Reals with no clear hope of working for FF
2. Some information theoretical arguments suggest that analog results for FF should be true: with very few observations it should be possible a unique matrix recovery.
3. It has applications to Coding Theory.
4. The FF case behaves differently than for Reals.

ACI-matrices might be an effective way to tackle minRank (and maxRank) completion problems in FF.

If you apply elementary row operation to a partial matrix you obtain...

$$\begin{bmatrix} x & z & \mathbf{1} \\ y & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & t \end{bmatrix} \underset{\substack{r_2 \to r_1 + r_2 \\ r_3 \to 2\, r_3}}{\longrightarrow} \begin{bmatrix} x & z & 1 \\ x+y & z+1 & 2 \\ 2 & 2 & 2t \end{bmatrix}$$

We “define” ACI-matrices as:

matrices that <u>seem</u> to be obtained by applying elementary row operation to a partial matrix

ACI-matrices Definition

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Definition (Brualdi, Huang, Zhan 2010)

ACI-matrices

- entries are either constants or polynomials of degree one
- no indeterminate appears in two different columns.

ACI-matrices are closed under

1. elementary row operations
2. column permutations

A, B are **equivalent** if you can tranform A into B by 1 and 2

Square

Brualdi, Huang and Zhan [2010], Borobia and Canogar [2012]

Let A be a $n \times n$ ACI-matrix. Then

$$\text{minRank}(A) = n \iff A \text{ is equivalent to an ACI-matrix of type } \begin{bmatrix} 1 & * & * \\ & \ddots & * \\ 0 & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ x & 0 & 1 & 1 \\ y & z & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & x \\ 1 & z & 0 & y \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & x-1 \\ 0 & z-1 & -1 & y-1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & z-1 & -1 & y-1 \\ 0 & -1 & 0 & x-1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & z-1 & y-1 \\ 0 & 0 & -1 & x-1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1-z & 1-y \\ 0 & 0 & 1 & 1-x \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rectangular

Huang and Zhan [2011]

Let A be a $m \times n$ ACI-matrix **over a field** $\mathbb{F}$ **with** $\|\mathbb{F}\| \geq m > n$. Then

$$\text{minRank}(A) = n \iff A \text{ is equivalent to an ACI-matrix of type } \begin{bmatrix} * & * & * \\ * & * & * \\ 1 & * & * \\ & \ddots & * \\ 0 & & 1 \end{bmatrix}$$

FF work differently: the restriction on the field can not be dropped.

Example: A is a 4×3 ACI-matrix over $\mathbb{F}_2$ with $\text{minRank}(A) = 3$ but can not be put into upper triangular form:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & x \\ y & 0 & 1 \\ 0 & z & 1 \end{bmatrix}$$

When $m < n$ we have the same phenomenon.

We have developed an algorithm to compute the maxRank for any ACI-matrix.

Definition

A is **full-maxRank** if it has a completion which is full rank

If the input A is not full-maxRank then the output is an equivalent ACI-Matrix

$*$	A_{12}
A_{21}	0

with a zero block so big that A_{12} and A_{21} are squashed,
and A_{12} and A_{21} are full-maxRank, and

$$\text{maxRank} = \text{maxRank}(A_{12}) + \text{maxRank}(A_{21}) = \#rows(A_{12}) + \#cols(A_{21}).$$

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Thank You!